Reduced Free Products of Unital AH Algebras and Blackadar and Kirchberg's MF Algebras

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Abstract: In the paper, we prove that reduced free products of unital AH algebras with respect to given faithful tracial states, in the sense of Voiculescu, are Blackadar and Kirhcberg's MF algebras. We also show that the reduced free products of unital AH algebras with respect to given faithful tracial states, under mild conditions, are not quasidiagonal. Therefore we conclude, for a large class of AH algebras, the Brown-Douglas-Fillmore extension semigroups of the reduced free products of these AH algebras with respect to given faithful tracial states are not groups.

Our result is based on Haagerup and Thorbjørsen's work on the reduced C*-algebras of free groups.

Keywords: MF algebras, Reduced free products, BDF semigroups

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1. Introduction

The BDF theory was developed by Brown, Douglas and Fillmore in 1977 in [**8**]. In order to classify essentially normal operators, they introduced an important invariant, $Ext(\mathcal{A})$ (the BDF extension semigroup), for a unital separable C*-algebra $\mathcal{A}$. Among many other things they proved in [**8**] that $Ext(C(X))$ is a group when X is a compact metric space. Later, Choi and Effros [**12**] showed that $Ext(\mathcal{A})$ is a group if $\mathcal{A}$ is a unital separable nuclear C*-algebra. By a result of Voiculescu, we know that the semigroup $Ext(\mathcal{A})$ always has a unit if $\mathcal{A}$ is a unital separable C*-algebra.

Anderson [**1**] provided the first example of a unital separable C*-algebra $\mathcal{A}$ such that $Ext(\mathcal{A})$ is not a group. Using Kazhdan's property T for groups, Wassermann gave other examples of unital separable C*-algebras $\mathcal{A}$ such that $Ext(\mathcal{A})$ is not a group in [**32**]. In [**23**], Kirchberg provided more examples of unital separable C*-algebras whose BDF extension semigroups are not groups by showing that the following result: A C*-algebra $\mathcal{A}$ has the local lifting property if and only if $Ext(S(\mathcal{A}))$ is a group, where $S(\mathcal{A})$ denotes the unitization of $C_0(\mathbb{R}) \otimes_{min} \mathcal{A}$.

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Ever since Anderson's example in [**1**], it has been an open problem whether $Ext(C_r^*(F_2))$, the BDF extension semigroup of the reduced C*-algebra of free group F_2, is a group. This problem was studied by many mathematician (see [**27**]) and finally settled down in the negative by Haagerup and Thorbjørnsen [**17**] using powerful tools developed from Voiculescu's free probability theory and random matrix theory. Their result that $Ext(C_r^*(F_2))$ is not a group follows from a combination of Voiculescu's result in [**27**] and their striking work on showing that $C_r^*(F_2)$ can be embedded into $\prod_k \mathcal{M}_{n_k}(\mathbb{C})/\sum_k \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$.

If a separable C*-algebra $\mathcal{A}$ can be embedded into $\prod_k \mathcal{M}_{n_k}(\mathbb{C})/\sum_k \mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$, then such C*-algebra $\mathcal{A}$ is call an MF algebra. The concept of MF algebra was introduced by Blackadar and Kirchberg in [**5**] in order to study the classification problem of C*-algebras. Many properties of MF algebras were discussed there. Using the concept of MF algebras, Brown in [**9**] (see also [**17**]) generalized Voiculescu's result in [**27**] as follows: *If a unital separable C*-algebra $\mathcal{A}$ is an MF algebra but not a quasidiagonal C*-algebra, then $Ext(\mathcal{A})$ is not a group.* Note by a result of Rosenberg, $C_r^*(F_2)$ is not quasidiagonal. Now Haagerup and Thorbjørnsen's work can be restated as follows: *$C_r^*(F_2)$ is an MF algebra and $Ext(C_r^*(F_2))$ is not a group.*

The concept of reduced free products of unital C*-algebras with respect to given states was provided by Voiculescu in the context of his free probability theory [**31**]. This concept plays an important role in the recent study of C*-algebras (for example see [**13**], [**14**], [**15**]). Assume that $(\mathcal{A}, \tau_{\mathcal{A}})$ and $(\mathcal{B}, \tau_{\mathcal{B}})$ are unital C*-algebras with faithful tracial states $\tau_{\mathcal{A}}$, and $\tau_{\mathcal{B}}$ respectively. In [**31**], Voiculescu introduced the reduced free product $(\mathcal{A}, \tau_{\mathcal{A}}) *_{red} (\mathcal{B}, \tau_{\mathcal{B}})$ of $(\mathcal{A}, \tau_{\mathcal{A}})$ and $(\mathcal{B}, \tau_{\mathcal{B}})$. A quick fact from the definition of reduced free product of C*-algebras is the following statement:

$$(C_r^*(F_2), \tau_{F_2}) = (C_r^*(\mathbb{Z}), \tau_{\mathbb{Z}}) *_{red} (C_r^*(\mathbb{Z}), \tau_{\mathbb{Z}}),$$

where, for a discrete countable group G, we let $C_r^*(G)$ be the reduced C*-algebra of group G and τ_G the canonical tracial state induced by the left regular representation λ, i.e. $\tau_G(\lambda(g)) = \langle \lambda(g)\delta_e, \delta_e \rangle$ for any g in G with δ_e the distinguished vector in $l^2(G)$.

In view of Haagerup and Thorbjørnsen's work on $C_r^*(F_2)$ and the preceding fact from the definition of reduced free products, one should naturally consider the following question:

> *What are necessary and sufficient conditions on unital separable C*-algebras $\mathcal{A}$ and $\mathcal{B}$ such that*
>
> $$Ext\big((\mathcal{A}, \tau_{\mathcal{A}}) *_{red} (\mathcal{B}, \tau_{\mathcal{B}})\big) \quad \textit{is not a group,}$$
>
> *where $\tau_{\mathcal{A}}$, and $\tau_{\mathcal{B}}$, are faithful tracial states of $\mathcal{A}$, and $\mathcal{B}$ respectively?*

This paper grows out in an attempt to understand Haagerup and Thorbjørnsen's result in [**17**] and search for answer to the preceding question. In fact, we are able to prove the following generalizations of Haagerup and Thorbjørnsen's result mentioned as above.

Theorem 4.2.1: *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital separable AH algebras with faithful tracial states τ_1, and τ_2 respectively. If $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy Avitzour's condition,* i.e. there are unitaries $u \in \mathcal{A}_1$ and $v, w \in \mathcal{A}_2$ such that $\tau_1(u) = \tau_2(v) = \tau_2(w) = \tau_2(w^*v) = 0$, *then*

$$Ext\big((\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)\big) \quad \textit{is not a group.}$$

Recall a unital separable C*-algebra $\mathcal{A}$ is an approximately homogeneous (AH) C*-algebra if $\mathcal{A}$ is an inductive limit of a sequence of homogeneous C*-algebras (see [**4**]). Obviously, all AF algebras, AI algebras and A$\mathbb{T}$ algebras are AH algebras.

Theorem 4.2.2: *Let $\mathcal{A}$ and $\mathcal{B} \neq \mathbb{C}$ be separable unital AH algebras with faithful traces ϕ, and ψ respectively. If $\mathcal{A}$ is partially diffuse in the sense of Definition 4.1.1, then*

$$Ext\big((\mathcal{A},\phi) *_{red} (\mathcal{B},\psi)\big) \quad \textit{is not a group.}$$

Theorem 4.2.3: *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital separable AF algebras with faithful tracial states ϕ, and ψ respectively. If $dim_{\mathbb{C}}\mathcal{A} \geq 2$ and $dim_{\mathbb{C}}\mathcal{B} \geq 3$, then*

$$Ext\big((\mathcal{A},\phi) *_{red} (\mathcal{B},\psi)\big) \quad \textit{is not a group.}$$

Using these results, one can easily produce new examples of unital separable C*-algebras whose BDF extension semigroups are not groups. For example, *Let $\mathcal{A}$ and $\mathcal{B}$ be irrational C*-algebras, or UHF algebras, with faithful traces ϕ, and ψ respectively. Then $Ext\big((\mathcal{A},\phi)*_{red}(\mathcal{B},\psi)\big)$ is not a group.* Combining with results from [**21**] and [**22**], one obtains more examples of unital separable C*-algebras whose BDF extension semigroups are not groups.

One crucial step in proving Theorem 4.2.1, Theorem 4.2.2 and Theorem 4.2.3 is our following result on Blackadar and Kirchberg's MF algebra, whose proof is based on Haagerup and Thorbjørnsen's result that $C_r^*(F_2)$ is an MF algebra.

Theorem 3.3.3: *Suppose that $\mathcal{A}_i$, $i = 1, \ldots, n$, is a family of unital separable AH algebras with faithful tracial states τ_i, $i = 1, \ldots, n$. Then*

$$(\mathcal{A}_1, \tau_1) *_{red} \cdots *_{red} (\mathcal{A}_n, \tau_n)$$

is an MF algebra.

Blackadar and Kirchberg's MF algebra is also closely connected to Voiculescu's topological free entropy dimension. In [**30**], for a family of self-adjoint elements $x_1, \ldots, x_n$ in a unital C*-algebra $\mathcal{A}$, Voiculescu introduced the notion of topological free entropy dimension of $x_1, \ldots, x_n$. In the definition of topological free entropy dimension, it requires that the C*-subalgebra generated by $x_1, \ldots, x_n$ in $\mathcal{A}$ is an MF algebra. Applying Theorem 3.3.3, we obtain the following result on Voiculescu's topological free entropy dimension:

Corollary 3.3.1: *Suppose that $(\mathcal{A}, \tau)$ is a C*-free probability space. Let $x_1, \ldots, x_n$ be a family of self-adjoint elements in $\mathcal{A}$ such that $x_1, \ldots, x_n$ are free with respect to τ. Then*

$$\delta_{top}(x_1, \ldots, x_n) \geq 0,$$

where $\delta_{top}(x_1, \ldots, x_n)$ is the Voiculescu's topological free entropy dimension.

The organization of the paper is as follows. In section 2, we introduce some notation and basic concepts needed in the later sections. In section 3, we start with Haagerup and Thorbjørnsen's result on $C_r^*(F_n)$ and show that the reduced free products of finite dimensional C*-algebras with respect to given tracial states are MF algebras. Then we conclude that the reduced free products of AH algebras with respect to given tracial states are MF algebras. In section 4, we show that the reduced free products of unital C*-algebras, under mild conditions, are non-quasidiagonal.

Combining the results from section 3, we reach our conclusions on reduced free products of unital AH algebras whose BDF extension semigroups are not groups in section 4. In section 5, we further discuss the reduced free products of some tensor products of unital C*-algebras, which are not covered in section 3.

2. Notation and Preliminaries

In this section, we will recall some basic facts on C*-algebras and introduce some lemmas that will be needed in the later sections.

2.1. Gram-Schmidt orthogonalization. Suppose $\mathcal{H}$ is a complex Hilbert space and $\langle \cdot, \cdot \rangle$ is an inner product on $\mathcal{H}$. Let $\{y_m\}_{m=1}^N$ be a family of linearly independent vectors in $\mathcal{H}$, where N is a positive integer. Let, for each $1 \le m \le N$,

$$P_m(y_1, \dots, y_N) = \begin{vmatrix} \langle y_1, y_1 \rangle & \langle y_2, y_1 \rangle & \cdots & \langle y_m, y_1 \rangle \\ \langle y_1, y_2 \rangle & \langle y_2, y_2 \rangle & \cdots & \langle y_m, y_2 \rangle \\ & & \cdots & \\ \langle y_1, y_{m-1} \rangle & \langle y_2, y_{m-1} \rangle & \cdots & \langle y_m, y_{m-1} \rangle \\ y_1 & y_2 & \cdots & y_m \end{vmatrix}.$$

Since $y_1, \dots, y_N$ are linearly independent, each $P_m(y_1, \dots, y_N) \neq 0$. We let

$$p_m(y_1, \dots, y_N) = \frac{P_m(y_1, \dots, y_N)}{\langle P_m(y_1, \dots, y_N), P_m(y_1, \dots, y_N) \rangle^{1/2}}, \qquad for\ 1 \le m \le N.$$

Then we have the following statement.

LEMMA 2.1.1. *Assume $y_1, \dots, y_N$ is a family of linearly independent vectors in $\mathcal{H}$ and $p_m(y_1, \dots, y_N)$ for $1 \le m \le N$ is defined as above. Then $\{p_m(y_1, \dots, y_N)\}_{m=1}^N$ forms an orthonormal basis for the closed subspace spanned by $y_1, \dots, y_N$ in $\mathcal{H}$.*

2.2. Reduced crossed products of C*-algebras by groups. Assume that $\mathcal{A}$ is a separable unital C*-algebra and G is a discrete countable group. Let α be a homomorphism from G into $Aut(\mathcal{A})$. Then we can define the reduced crossed product, $\mathcal{A} \rtimes_{\alpha,r} G$, of $\mathcal{A}$ by the action α of G as follows. Let $\rho : \mathcal{A} \to B(\mathcal{H})$ be a faithful $*$-representation of $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$. Let $l^2(G)$ be the Hilbert space associated to G with an orthonormal basis $\{e_g\}_{g \in G}$. Let $\lambda : G \to B(l^2(G))$ be the left regular representation of G on the Hilbert space $l^2(G)$. Then we let $\mathcal{K} = \mathcal{H} \otimes l^2(G)$. And we introduce a representation σ of $\mathcal{A}$ and G on $\mathcal{K}$ by the following:

$$\sigma(g) = 1 \otimes \lambda(g), \qquad \forall\ g \in G$$
$$\sigma(x)(\xi \otimes e_g) = (\alpha^{-1}(g)(x)\xi) \otimes e_g, \quad \forall\ x \in \mathcal{A},\ \forall\ \xi \in \mathcal{H}, \quad g \in G.$$

Then the C*-algebra generated by $\{\sigma(g)\}_{g \in G}$ and $\{\sigma(x)\}_{x \in \mathcal{A}}$ in $B(\mathcal{K})$ is called the reduced crossed product of $\mathcal{A}$ by G, and is denoted by $\mathcal{A} \rtimes_{\alpha,r} G$. We know that the reduced crossed product $\mathcal{A} \rtimes_{\alpha,r} G$ does not depend on the choice of the faithful $*$-representation ρ of $\mathcal{A}$.

2.3. Reduced free products of unital C*-algebras. The concept of reduced free products of unital C*-algebras was introduced by Voiculescu in the context of his free probability theory. (see [**31**], also [**3**])

Assume that $\mathcal{A}_i$, $i = 1, 2$, is a separable unital C*-algebra with a state τ_i. For each $i = 1, 2$, let $(\mathcal{H}_i, \xi_i, \pi_i)$ be the GNS representation of $\mathcal{A}_i$ on the Hilbert space $\mathcal{H}_i$ such that (i) $\tau_i(x_i) = \langle \pi_i(x_i)\xi_i, \xi_i\rangle$ for all $x_i \in \mathcal{A}_i$ and (ii) $\mathcal{H}_i = \overline{\{\pi_i(x_i)\xi_i \mid \ x_i \in \mathcal{A}_i\}}$.

Let $\overset{\circ}{\mathcal{H}}_i = \mathcal{H}_i \ominus \mathbb{C}\xi_i$ for $i = 1, 2$. The Hilbert space free product of $(\mathcal{H}_1, \xi_1)$ and $(\mathcal{H}_2, \xi_2)$ is given by

$$\mathcal{H} = (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2) = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \left(\bigoplus_{j_1 \neq j_2 \neq \cdots \neq j_n} \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n} \right),$$

where ξ is the distinguished unit vector in $\mathcal{H}$. Let, for $i = 1, 2$,

$$\mathcal{H}(i) = \mathbb{C}\xi \oplus \bigoplus_{n\geq 1} \left(\bigoplus_{i \neq j_1 \neq j_2 \neq \cdots \neq j_n} \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n} \right).$$

We can define unitary operators $V_i : \mathcal{H}_i \otimes \mathcal{H}(i) \to \mathcal{H}$ as follows:

$$\begin{gathered}
\xi_i \otimes \xi \mapsto \xi \\
\overset{\circ}{\mathcal{H}}_i \otimes \xi \mapsto \overset{\circ}{\mathcal{H}}_i \\
\xi_i \otimes (\overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}) \mapsto \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n} \\
\overset{\circ}{\mathcal{H}}_i \otimes (\overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}) \mapsto \overset{\circ}{\mathcal{H}}_i \otimes \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}
\end{gathered}$$

Let λ_i be the representation of $\mathcal{A}_i$ on $\mathcal{H}$ given by

$$\lambda_i(x) = V_i(\pi_i(x) \otimes I_{\mathcal{H}(i)})V_i^*, \qquad \forall\ x \in \mathcal{A}_i.$$

Then the reduced free product of $(\mathcal{A}_1, \tau_1)$ and $(\mathcal{A}_2, \tau_2)$, or the reduced free product of $\mathcal{A}_1$ and $\mathcal{A}_2$ with respect to τ_1 and τ_2, is the C*-algebra generated by $\lambda_1(\mathcal{A}_1)$ and $\lambda_2(\mathcal{A}_2)$ in $B(\mathcal{H})$, and is denoted by

$$(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2).$$

Moreover, the free product state $\tau = \tau_1 * \tau_2$ on $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$, given by $\tau(x) = \langle x\xi, \xi\rangle$, is a faithful tracial state if both τ_1 and τ_2 are faithful tracial states on $\mathcal{A}_1$, and $\mathcal{A}_2$ respectively.

Remark 2.3.1. *Suppose that $\mathcal{A}_1$, and $\mathcal{A}_2$ are unital C*-algebras with faithful tracial states τ_1, and τ_2 respectively. Suppose that $I_{\mathcal{A}_1} \in \mathcal{B}_1$, and $I_{\mathcal{A}_2} \in \mathcal{B}_2$, are unital C*-subalgebras of $\mathcal{A}_1$, and $\mathcal{A}_2$ respectively. Then there is an embedding*

$$(\mathcal{B}_1, \tau_1|_{\mathcal{B}_1}) *_{red} (\mathcal{B}_2, \tau_2|_{\mathcal{B}_2}) \subseteq (\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2).$$

2.4. Blackadar and Kirchberg's MF algebras. Recall the definition of Blackadar and Kirchberg's MF algebras ([**5**]) as follows.

Definition 2.4.1. *A separable C*-algebra $\mathcal{A}$ is called an MF algebra if there is an embedding from $\mathcal{A}$ into $\prod_{k=1}^{\infty}\mathcal{M}_{n_k}(\mathbb{C})/\sum_{k=1}^{\infty}\mathcal{M}_{n_k}(\mathbb{C})$ for a sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ where $\mathcal{M}_{n_k}(\mathbb{C})$ is the $n_k \times n_k$ complex matrix algebra.*

We will need the following lemma in the later sections.

Lemma 2.4.1. *Let $\mathcal{A}$ be a unital separable C*-algebra. Then the following are equivalent.*

(i) *$\mathcal{A}$ is an MF algebra;*
(ii) *For any family of self-adjoint elements $x_1, \ldots, x_n$ in $\mathcal{A}$, any $\epsilon > 0$, and any family of noncommutative polynomials $P_1, \ldots, P_r$ in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$, there are a positive integer k and a family of self-adjoint matrices*

$$A_1, \ldots, A_n \qquad in \;\; \mathcal{M}_k(\mathbb{C})$$

such that

$$\max_{1\le j\le r} |\|P_j(x_1, \ldots, x_n)\| - \|P_j(A_1, \ldots, A_n)\|| \le \epsilon.$$

Proof. Note a separable C*-algebra $\mathcal{A}$ is an MF algebra if and only if every finitely generated C*-subalgebra of $\mathcal{A}$ is an MF algebra (see Corollary 3.4.4 in [**5**]). The rest follows from Theorem 5.2 in [**21**]. □

Remark 2.4.1. *A separable C*-subalgebra of an MF algebra is also an MF algebra.*

3. Reduced Free Products of AH Algebras

In this section, we are going to show that reduced free products of unital AH algebras with respect to given faithful tracial states are MF algebras. First, we need to consider GNS representation of a finite dimensional C*-algebra.

3.1. GNS representation of a finite dimensional C*-algebra. Suppose that $\mathcal{B}$ is a finite dimensional C*-algebra and ψ is a faithful tracial state of $\mathcal{B}$.

Let $d = dim_{\mathbb{C}}\mathcal{B}$, the complex dimension of $\mathcal{B}$. Then there is a family of elements $1, b_1, \ldots, b_{d-1}$ in $\mathcal{B}$ that forms a basis of $\mathcal{B}$, where 1 is the identity of $\mathcal{B}$. Note ψ is a faithful tracial state of $\mathcal{B}$. We can introduce an inner product on $\mathcal{B}$ as follows.

$$\langle x, y\rangle = \psi(y^*x), \qquad \forall\; x, y \in \mathcal{B}.$$

By the Gram-Schmidt orthogonalization in Section 2.1, for the basis $1, b_1, \ldots, b_{d-1}$ of $\mathcal{B}$, we let

$$P_1(1, b_1, \ldots, b_{d-1} : \psi) = 1$$

and

$$P_m(1, b_1, \ldots, b_{d-1} : \psi) = \begin{vmatrix} 1 & \psi(b_1) & \cdots & \psi(b_{m-1}) \\ \psi(b_1^*) & \psi(b_1^* b_1) & \cdots & \psi(b_1^* b_{m-1}) \\ & & \cdots & \\ \psi(b_{m-2}^*) & \psi(b_{m-2}^* b_1) & \cdots & \psi(b_{m-2}^* b_{m-1}) \\ 1 & b_1 & \cdots & b_{m-1} \end{vmatrix}, \qquad 2 \le m \le d;$$

and

$$p_m(1, b_1, \ldots, b_{d-1} : \psi) = \frac{P_m(1, b_1, \ldots, b_{d-1} : \psi)}{\left(\psi(P_m(1, b_1, \ldots, b_{d-1} : \psi)^* P_m(1, b_1, \ldots, b_{d-1} : \psi))\right)^{1/2}}, \qquad 1 \le m \le d.$$

Then

$$1 = p_1(1, b_1, \ldots, b_{d-1} : \psi),\ p_2(1, b_1, \ldots, b_{d-1} : \psi),\ \ldots,\ p_d(1, b_1, \ldots, b_{d-1} : \psi)$$

forms an orthonormal basis of $\mathcal{B} = L^2(\mathcal{B}, \psi)$.

LEMMA 3.1.1. *Suppose that $\mathcal{B}$ is a finite dimensional C^*-algebra with a basis $1, b_1, \ldots, b_{d-1}$, where d is the complex dimension of $\mathcal{B}$. Suppose that ψ is a faithful tracial state of $\mathcal{B}$. Let*

$$1 = p_1(1, b_1, \ldots, b_{d-1} : \psi),\ p_2(1, b_1, \ldots, b_{d-1} : \psi),\ \ldots,\ p_d(1, b_1, \ldots, b_{d-1} : \psi)$$

be defined as above.

Let $\mathbb{C}^d$ be a d-dimensional complex Hilbert space with an orthonormal basis $e_1, \ldots, e_d$. Then there is a faithful unital $*$-representation $\rho_\psi : \mathcal{B} \to \mathcal{M}_d(\mathbb{C})$ of $\mathcal{B}$ on $\mathbb{C}^d$ such that

(i) *$(\rho_\psi, \mathbb{C}^d, e_1)$ is a GNS representation of $(\mathcal{B}, \psi)$, i.e.*
  (a) *$\psi(a) = \langle \rho_\psi(a) e_1, e_1 \rangle$ for all $a \in \mathcal{B}$.*
  (b) $\mathbb{C}^d = \overline{\{\rho_\psi(a) e_1 \mid a \in \mathcal{B}\}}$

(ii) *For each $1 \le i \le d-1$,*

$$\rho_\psi(b_i) = B_{i,\psi} = [b(s, t : i, \psi)]_{s,t=1}^d \in \mathcal{M}_d(\mathbb{C})$$

where $b(s, t : i, \psi)$, the (s,t)-th entry of the matrix $B_{i,\psi}$, is given by

$$b(s, t : i, \psi) = \psi(p_t(1, b_1, \ldots, b_{d-1} : \psi)^* b_i p_s(1, b_1, \ldots, b_{d-1} : \psi))$$

PROOF. Note that $\mathcal{B}$ is a finite dimensional C*-algebra with a faithful tracial state ψ. We can view $\mathcal{B} = L^2(\mathcal{B}, \psi)$ as a Hilbert space with the inner product induced from ψ. Thus $\mathcal{B} = L^2(\mathcal{B}, \psi)$ is isomorphic to $\mathbb{C}^d$ as a Hilbert space. By the explanation preceding the lemma, we can introduce a unitary $U : L^2(\mathcal{B}, \psi) \to \mathbb{C}^d$ by mapping

$$p_m(1, b_1, \ldots, b_{d-1} : \psi) \mapsto e_m, \qquad \forall\ 1 \le m \le d.$$

Apparently, such U induces a faithful unital $*$-representation $\rho_\psi : \mathcal{B} \to \mathcal{M}_d(\mathbb{C})$ by

$$\rho_\psi(b) = U b U^*, \qquad \forall\ b \in \mathcal{B}.$$

Now it is easy to verify that $(\rho_\psi, \mathbb{C}^d, e_1)$ is a GNS representation of $(\mathcal{B}, \psi)$ satisfying (a) and (b). Moreover, for each $1 \le i \le d-1$,

$$\rho_\psi(b_i) = B_{i,\psi} = [b(s,t:i,\psi)]_{s,t=1}^d \in \mathcal{M}_d(\mathbb{C})$$

satisfying

$$b(s,t:i,\psi) = \psi(p_t(1,b_1,\dots,b_{d-1}:\psi)^* b_i p_s(1,b_1,\dots,b_{d-1}:\psi)).$$

This completes the proof. $\square$

Lemma 3.1.2. *Suppose that $\mathcal{B}$ is a finite dimensional C^*-algebra with a basis $1, b_1, \dots, b_{d-1}$, where d is the complex dimension of $\mathcal{B}$. Suppose that $\{\tau, \tau_\gamma\}_{\gamma=1}^\infty$ is a family of faithful tracial states of $\mathcal{B}$ satisfying*

$$\lim_{\gamma\to\infty} \tau_\gamma(b) = \tau(b) \qquad \forall\ b \in \mathcal{B}.$$

Let $\mathbb{C}^d$ be a d-dimensional complex Hilbert space with an orthonormal basis $e_1, \dots, e_d$. Then there is a sequence of faithful unital $*$-representations $\rho_\tau, \rho_{\tau_\gamma} : \mathcal{B} \to \mathcal{M}_d(\mathbb{C})$ of $\mathcal{B}$ on $\mathbb{C}^d$ for $\gamma = 1, 2 \dots$ such that

(i) *$(\rho_\tau, \mathbb{C}^d, e_1)$ and $(\rho_{\tau_\gamma}, \mathbb{C}^d, e_1)$ are GNS representations of $(\mathcal{B}, \tau)$, and $(\mathcal{B}, \tau_\gamma)$ respectively.*
(ii) *For each $1 \le i \le d-1$,*

$$\lim_{\gamma\to\infty} \|\rho_{\tau_\gamma}(b_i) - \rho_\tau(b_i)\| = 0$$

Proof. Note $\mathcal{B}$ is a finite dimensional C*-algebras with a basis $1, b_1, \dots, b_{d-1}$ and $\{\tau, \tau_\gamma\}_{\gamma=1}^\infty$ is a family of faithful tracial states of $\mathcal{B}$. $\mathbb{C}^d$ is a d-dimensional complex Hilbert space with an orthonormal basis $e_1, \dots, e_d$. By Lemma 3.1.1, there is a sequence of faithful unital $*$-representations $\rho_\tau, \rho_{\tau_\gamma} : \mathcal{B} \to \mathcal{M}_d(\mathbb{C})$ of $\mathcal{B}$ on $\mathbb{C}^d$ for $\gamma = 1, 2 \dots$ such that

(iii) $(\rho_\tau, \mathbb{C}^d, e_1)$ and $(\rho_{\tau_\gamma}, \mathbb{C}^d, e_1)$ are GNS representations of $(\mathcal{B}, \tau)$, and $(\mathcal{B}, \tau_\gamma)$ respectively.
(iv) Moreover, for each $1 \le i \le d-1$,

$$\rho_\tau(b_i) = B_{i,\tau} = [b(s,t:i,\tau)]_{s,t=1}^d \in \mathcal{M}_d(\mathbb{C})$$

satisfying

$$b(s,t:i,\tau) = \tau(p_t(1,b_1,\dots,b_{d-1}:\tau)^* b_i p_s(1,b_1,\dots,b_{d-1}:\tau));$$

and, for $\gamma = 1, 2, \dots$

$$\rho_{\tau_\gamma}(b_i) = B_{i,\tau_\gamma} = [b(s,t:i,\tau_\gamma)]_{s,t=1}^d \in \mathcal{M}_d(\mathbb{C})$$

satisfying

$$b(s,t:i,\tau_\gamma) = \tau_\gamma(p_t(1,b_1,\dots,b_{d-1}:\tau_\gamma)^* b_i p_s(1,b_1,\dots,b_{d-1}:\tau_\gamma)).$$

Since

$$\lim_{\gamma\to\infty} \tau_\gamma(b) = \tau(b), \qquad \forall\ b \in \mathcal{B},$$

by the choices of

$$1 = p_1(1,b_1,\dots,b_{d-1}:\tau), \quad \dots, \quad p_d(1,b_1,\dots,b_{d-1}:\tau)$$

and

$$1 = p_1(1, b_1, \ldots, b_{d-1} : \tau_\gamma), \quad \ldots, \quad p_d(1, b_1, \ldots, b_{d-1} : \tau_\gamma)$$

in the discussion before Lemma 3.1.1, we know that

$$\lim_{\gamma\to\infty} b(s, t : i, \tau_\gamma) = b(s, t : i, \tau).$$

It follows that

$$\lim_{\gamma\to\infty} \|\rho_{\tau_\gamma}(b_i) - \rho_\tau(b_i)\| \le \lim_{\gamma\to\infty} d^2 \left(\max_{1\le s,t\le d} |b(s, t : i, \tau_\gamma) - b(s, t : i, \tau)| \right) = 0, \quad \forall\ 1 \le i \le d-1.$$

□

DEFINITION 3.1.1. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are separable unital C*-algebras. Let $\epsilon > 0$ be a positive number. Suppose that $x_1, \ldots, x_n$ is a family of elements in $\mathcal{A}$. Then we call*

$$\{x_1, \ldots, x_n\} \subseteq_\epsilon \mathcal{B}$$

if the following holds:

> *There are (i) $y_1, \ldots, y_n$ in $\mathcal{B}$ and (ii) unital faithful *-representations $\rho_1 : \mathcal{A} \to B(\mathcal{H})$ and $\rho_2 : \mathcal{B} \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ such that*
>
> $$\max_{1\le i\le n} \|\rho_1(x_i) - \rho_2(y_i)\| \le \epsilon.$$

LEMMA 3.1.3. *Suppose that $\mathcal{A}$ is a separable unital C*-algebra with a faithful tracial state ψ. Suppose that $\mathcal{B}$ is a finite dimensional C*-algebra with a family $\{\tau, \tau_\gamma\}_{\gamma=1}^\infty$ of faithful tracial states of $\mathcal{B}$ such that*

$$\lim_{\gamma\to\infty} \tau_\gamma(b) = \tau(b), \qquad \forall\ b \in \mathcal{B}.$$

Suppose that $x_1, \ldots, x_n$ is a family of elements in $(\mathcal{A}, \psi) *_{red} (\mathcal{B}, \tau)$. Then, for any $\epsilon > 0$, there is a $\gamma_0 > 0$ such that

$$\{x_1, \ldots, x_n\} \subseteq_\epsilon (\mathcal{A}, \psi) *_{red} (\mathcal{B}, \tau_\gamma), \qquad \forall\ \gamma > \gamma_0.$$

PROOF. Note that $\mathcal{B}$ is a finite dimensional C*-algebra. Assume that $1, b_1, \ldots, b_{d-1}$ is a basis of $\mathcal{B}$, where d is the complex dimension of $\mathcal{B}$. Let $\mathbb{C}^d$ be a d-dimensional complex Hilbert space with an orthonormal basis $e_1, \ldots, e_d$. By Lemma 3.1.2, there is a sequence of faithful unital $*$-representations $\rho_\tau, \rho_{\tau_\gamma} : \mathcal{B} \to \mathcal{M}_d(\mathbb{C})$ of $\mathcal{B}$ on $\mathbb{C}^d$ for $\gamma = 1, 2 \ldots$ such that

(i) $(\rho_\tau, \mathbb{C}^d, e_1)$ and $(\rho_{\tau_\gamma}, \mathbb{C}^d, e_1)$ are GNS representations of $(\mathcal{B}, \tau)$, and $(\mathcal{B}, \tau_\gamma)$ respectively.
(ii) For each $1 \le i \le d-1$,

$$\lim_{\gamma\to\infty} \|\rho_{\tau_\gamma}(b_i) - \rho_\tau(b_i)\| = 0 \tag{3.1.1}$$

Let $(\pi, \mathcal{H}_1, \xi_1)$ be the GNS representation of $(\mathcal{A}, \psi)$ on a Hilbert space $\mathcal{H}_1$ such that ξ_1 is cyclic for $\pi(\mathcal{A})$ and $\psi(a) = \langle \pi(a)\xi_1, \xi_1 \rangle$ for all a in $\mathcal{A}$.

Let $(\mathcal{H}_2, \xi_2) = (\mathbb{C}^d, e_1)$ and $\mathcal{H}$ be the free product of Hilbert spaces $(\mathcal{H}_1, \xi_1)$ and $(\mathcal{H}_2, \xi_2)$ as in Section 2.3. Let $\overset{\circ}{\mathcal{H}}_i$ and $\mathcal{H}(i)$ be defined as in Section 2.3 for $i = 1, 2$ and V_1, V_2 be the unitary

operators as defined in Section 2.3. Let λ be the representation of $\mathcal{A}$ and $\mathcal{B}$ on the Hilbert space $\mathcal{H}$ defined as follows:

$$\lambda(a) = V_1(\pi(a) \otimes I_{\mathcal{H}(1)})V_1^*, \qquad \forall\ a \in \mathcal{A}; \tag{3.1.2}$$

$$\lambda(b) = V_2(\rho_\tau(b) \otimes I_{\mathcal{H}(2)})V_2^*, \qquad \forall\ b \in \mathcal{B}; \tag{3.1.3}$$

Let λ_γ, $\gamma = 1, 2, \ldots$, be a sequence of representations of $\mathcal{A}$ and $\mathcal{B}$ on the Hilbert space $\mathcal{H}$ defined as follows:

$$\lambda_\gamma(a) = V_1(\pi(a) \otimes I_{\mathcal{H}(1)})V_1^*, \qquad \forall\ a \in \mathcal{A}; \tag{3.1.4}$$

$$\lambda_\gamma(b) = V_2(\rho_\gamma(b) \otimes I_{\mathcal{H}(2)})V_2^*, \qquad \forall\ b \in \mathcal{B}; \tag{3.1.5}$$

Then by the definition of reduced free product in Section 2.3, we know that

(a) $(\mathcal{A}, \psi) *_{red} (\mathcal{B}, \tau)$ is the unital C*-subalgebra of $B(\mathcal{H})$ generated by

$$\{\lambda(a) \mid a \in \mathcal{A}\} \cup \{\lambda(b) \mid b \in \mathcal{B}\};$$

(b) $(\mathcal{A}, \psi) *_{red} (\mathcal{B}, \tau_\gamma)$ is the unital C*-subalgebra of $B(\mathcal{H})$ generated by

$$\{\lambda_\gamma(a) \mid a \in \mathcal{A}\} \cup \{\lambda_\gamma(b) \mid b \in \mathcal{B}\}.$$

Moreover, by (3.1.2) and (3.1.4), we know that

$$\lambda(a) = \lambda_\gamma(a), \qquad \forall\ a \in \mathcal{A} \tag{3.1.6}$$

By (3.1.1), (3.1.3) and (3.1.5), we know that, for $1 \le i \le d-1$,

$$\lim_{\gamma\to\infty} \|\lambda(b_i) - \lambda_\gamma(b_i)\| = \lim_{\gamma\to\infty} \|V_2(\rho_\gamma(b_i) \otimes I_{\mathcal{H}(2)})V_2^* - V_2(\rho_\tau(b_i) \otimes I_{\mathcal{H}(2)})V_2^*\| = 0. \tag{3.1.7}$$

Since $x_1, \ldots, x_n$ are in $(\mathcal{A}, \psi) *_{red} (\mathcal{B}, \tau)$, there are elements $a_1, \ldots, a_N$ in $\mathcal{A}$ and noncommutative polynomials $P_1, \ldots, P_n$ such that, for all $1 \le j \le n$,

$$\|x_j - P_j(\lambda(a_1), \ldots, \lambda(a_N), \lambda(b_1), \ldots, \lambda(b_{d-1}))\| \le \epsilon/3.$$

On the other hand, by (3.1.6) and (3.1.7), we know when γ is large enough, for all $1 \le j \le n$.

$$\|P_j(\lambda(a_1), \ldots, \lambda(a_N), \lambda(b_1), \ldots, \lambda(b_{d-1})) - P_j(\lambda_\gamma(a_1), \ldots, \lambda_\gamma(a_N), \lambda_\gamma(b_1), \ldots, \lambda_\gamma(b_{d-1}))\| \le \epsilon/3.$$

Therefore, when γ is large enough,

$$\|x_j - P_j(\lambda_\gamma(a_1), \ldots, \lambda_\gamma(a_N), \lambda_\gamma(b_1), \ldots, \lambda_\gamma(b_{d-1}))\| \le \epsilon.$$

I.e. when γ is large enough, we have $\{x_1, \ldots, x_n\} \subseteq_\epsilon (\mathcal{A}, \psi) *_{red} (\mathcal{B}, \tau_\gamma)$. $\square$

3.2. Reduced free products of matrix algebras. In this subsection, we will show that the reduced free products of matrix algebras with respect to given tracial states are MF algebras. Recall the following remarkable result of Haagerup and Thorbjørnsen.

LEMMA 3.2.1 (Haagerup and Thorbjørnsen). *For all positive integer $n \ge 2$, $C_r^*(F_n)$ is an MF algebra, where F_n is the nonabelian free group on n generators and $C_r^*(F_n)$ is the reduced group C*-algebra of the free group F_n.*

The following result can be found in Thereom 4.1 in [**22**].

LEMMA 3.2.2. *Suppose $\mathcal{A}$ is a unital MF algebra and G is a finite group. Suppose that $\alpha : G \to Aut(\mathcal{A})$ is a homomorphism from G into $Aut(\mathcal{A})$. Then the reduced crossed product, $\mathcal{A} \rtimes_{\alpha,r} G$, of $\mathcal{A}$ by G is an MF algebra.*

PROOF. For the purpose of completeness, we sketch its proof here. In fact the proof follows directly from the definition of reduced crossed product in section 2.2.

Recall the definition of reduced crossed product as follows. Assume that $\mathcal{A}$ acts on a Hilbert space $\mathcal{H}$. Let $l^2(G)$ be the Hilbert space associated to the group G with an orthonormal basis $\{e_g\}_{g\in G}$ and λ be the left regular representation of G on $l^2(G)$. Let E_g be the rank one projection from $l^2(G)$ onto the vector e_g in $l^2(G)$. Let σ be the representation of $\mathcal{A}$ and G on $\mathcal{H} \otimes l^2(G)$ induced by the following mapping:

$$\sigma(g) = I_{\mathcal{H}} \otimes \lambda(g), \qquad \forall\ g \in G;$$
$$\sigma(x) = \sum_{g\in G} \alpha_g^{-1}(x) \otimes E_g, \qquad \forall\ x \in \mathcal{A}.$$

Then the reduced crossed product, $\mathcal{A} \rtimes_{\alpha,r} G$, of $\mathcal{A}$ by G is the C*-subalgebra generated by $\{\sigma(g)\}_{g\in G} \cup \{\sigma(x)\}_{x\in\mathcal{A}}$ in $B(\mathcal{H} \otimes l^2(G))$.

Note G is a finite group. Then $B(l^2(G)) \simeq \mathcal{M}_k(\mathbb{C})$ for some positive integer k. Moreover, for all $g \in G$ and $x \in \mathcal{A}$, $\sigma(g)$ and $\sigma(x)$ are in $\mathcal{A}\otimes B(l^2(G))$. Since $\mathcal{A}$ is an MF algebra, $\mathcal{A}\otimes B(l^2(G))$ is also an MF algebra (see Proposition 3.3.6 in [**5**]). It follows that $\mathcal{A} \rtimes_{\alpha,r} G$, as a C*-subalgebra of $\mathcal{A} \otimes B(l^2(G))$, is also an MF algebra. This completes the proof of the lemma. ☐

A quick corollary of the preceding lemma is the following statement.

COROLLARY 3.2.1. *For any positive integer $n \geq 2$, $C_r^*(\mathbb{Z}_n * F_n)$ is an MF algebra, where $\mathbb{Z}_n$ is the quotient group $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_n * F_n$ is the free product of group $\mathbb{Z}_n$ and the free group F_n.*

PROOF. Assume that u is a natural generator of $\mathbb{Z}_n$, i.e. $u^n = e$ and $u^j \neq e$ for all $1 \leq j < n$. Assume that $g_1, \ldots, g_n$ are the natural generators of F_n. Let α be an action of $\mathbb{Z}_n$ on F_n induced by the following mapping:

$$\alpha(u)(g_i) = g_{i+1} \quad for\ 1 \leq i \leq n-1, \qquad and \quad \alpha(u)(g_n) = g_1.$$

Let $h_i = g_1^i g_2^i \ldots g_n^i$ for $i = 1, 2, \ldots, n$ be elements in F_n. Then we observe that as elements in $F_n \rtimes_\alpha \mathbb{Z}_n$, the elements u, $h_1, \ldots, h_n$ are free in $F_n \rtimes_\alpha \mathbb{Z}_n$. In other words, $\mathbb{Z}_n * F_n$ can be viewed as a subgroup of $F_n \rtimes_\alpha \mathbb{Z}_n$. Therefore,

$$C_r^*(\mathbb{Z}_n * F_n) \subseteq C_r^*(F_n \rtimes_\alpha \mathbb{Z}_n) = C_r^*(F_n) \rtimes_{\alpha,r} \mathbb{Z}_n.$$

By Haagerup and Thorbjørnsen's result and Lemma 3.2.2, we know that $C_r^*(\mathbb{Z}_n * F_n)$ is an MF algebra. ☐

LEMMA 3.2.3. *For any $n \geq 2$, let τ_n be the normalized trace on $\mathcal{M}_n(\mathbb{C})$. Then*

$$(\mathcal{M}_n(\mathbb{C}), \tau_n) *_{red} (\mathcal{M}_n(\mathbb{C}), \tau_n)$$

is an MF algebra.

Proof. Assume that the group $\mathbb{Z}_n * F_n$ is generated by the natural generators u in $\mathbb{Z}$ and $g_1, \ldots, g_n$ in F_n. Let λ be the left regular representation of $\mathbb{Z}_n * F_n$ on the Hilbert space $l^2(\mathbb{Z}_n * F_n)$ with the cyclic and separating vector η_1. Thus $C_r^*(\mathbb{Z}_n * F_n)$ is generated by $\lambda(u)$ and $\lambda(g_1), \ldots, \lambda(g_n)$ in $B(l^2(\mathbb{Z}_n * F_n))$.

Assume the second copy of $\mathbb{Z}_n$ is generated by another natural generator v. Let γ in $\mathbb{C}$ be the n-th root of unit. Let $\alpha : \mathbb{Z}_n \to Aut(C_r^*(\mathbb{Z}_n * F_n))$ be a homomorphism from $\mathbb{Z}_n$ into $Aut(C_r^*(\mathbb{Z}_n * F_n))$ induced by the following mapping:

$$\begin{aligned} \alpha(v)(\lambda(u)) &= \gamma\lambda(u) \\ \alpha(v)(\lambda(g_i)) &= \lambda(g_{i+1}), \quad for\ 1 \le i \le n-1 \\ \alpha(v)(\lambda(g_n)) &= \lambda(g_1). \end{aligned}$$

Let $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$ be the reduced crossed product of $C_r^*(\mathbb{Z}_n * F_n)$ by the group $\mathbb{Z}_n$. Recall the definition of reduced crossed product of C*-algebras as in Section 2.2. Let $l^2(\mathbb{Z}_n)$ be the Hilbert space with an orthonormal basis $e_1, e_v, e_{v^2}, \ldots, e_{v^{n-1}}$. Let $\lambda : \mathbb{Z}_n \to B(l^2(\mathbb{Z}_n))$ be the left regular representation of $\mathbb{Z}_n$ on the Hilbert space $l^2(\mathbb{Z}_n)$ with the cyclic and separating vector e_1. Let $\mathcal{H} = l^2(\mathbb{Z}_n * F_n) \otimes l^2(\mathbb{Z}_n)$. Then we introduce representation σ of $\mathbb{Z}_n$ and $\mathbb{Z}_n * F_n$ on $\mathcal{H}$ as following

$$\begin{aligned} \sigma(g) &= I_{l^2(\mathbb{Z}_n * F_n)} \otimes \lambda(g), \qquad \forall\ \ g \in \mathbb{Z}_n \\ \sigma(h)(\xi \otimes e_{v^i}) &= (\alpha^{-1}(v^i)(\lambda(h))(\xi)) \otimes e_{v^i}, \qquad \forall\ h \in \mathbb{Z}_n * F_n,\ \forall\ 1 \le i \le n. \end{aligned}$$

Then $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$ is the C*-algebra generated by $\{\sigma(g), \sigma(h) \mid g \in \mathbb{Z}_n\ and\ h \in \mathbb{Z}_n * F_n\}$ in $B(\mathcal{H})$. And we have

$$\sigma(v)\sigma(u) = \gamma\sigma(u)\sigma(v).$$

Furthermore, there is a canonical faithful tracial state τ on $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$, which is defined by

$$\tau(x) = \langle x(\eta_1 \otimes e_1), \eta_1 \otimes e_1 \rangle, \qquad \forall\ x \in C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n.$$

Claim 3.2.1. *$\{\sigma(v), \sigma(u)\}$ and $\{\sigma(g_1)\}$ are free with respect to τ in $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$*

[Proof of Claim:] Note that $\{\sigma(u^i)\sigma(v^j) \mid 0 \le i, j \le n-1\}$ forms a basis for the C*-subalgebra generated by $\sigma(u)$ and $\sigma(v)$ in $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$. And $\{\sigma(g_1^t)\}_{t=-\infty}^{\infty}$ forms a basis for the C*-subalgebra generated by $\sigma(g_1)$ in $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$. Therefore to prove the claim it suffices to show the following: *For any positive integer r, nonzero integers $n_1, n_2, , \ldots, n_r$, and integers $m_1, k_1, \ldots, m_r, k_r$ with $0 \le m_i, k_i < n$ and $(m_i, k_i) \ne (0, 0)$ for $1 \le i \le r$, we have*

$$\tau(\sigma(g_1^{n_1})\sigma(u^{m_1})\sigma(v^{k_1}) \cdots \sigma(g_1^{n_r})\sigma(u^{m_r})\sigma(v^{k_r})) = 0.$$

Note that

$$\begin{aligned} \tau(\sigma(g_1^{n_1})\sigma(u^{m_1})\sigma(v^{k_1}) \cdots \sigma(g_1^{n_r})\sigma(u^{m_r})\sigma(v^{k_r})) &= \tau(\left(\sigma(v^{k_r})\sigma(g_1^{n_1})\sigma(v^{k_r})^*\right)\left(\sigma(v^{k_r})\sigma(u^{m_1})\sigma(v^{k_r})^*\right) \\ &\left(\sigma(v^{k_r+k_1})\sigma(g_1^{n_2})\sigma(v^{k_r+k_1})^*\right) \cdots \left(\sigma(v^{k_r+k_1+\cdots k_{r-1}})\sigma(u^{m_r})\sigma(v^{k_r+k_1+\cdots k_{r-1}})^*\right)\sigma(v^{k_r+k_1+\cdots k_{r-1}})) \end{aligned}$$

Thus it will be enough if we are able to show the following is true: *For any positive integer r, nonzero integers $n_1, n_2, , \ldots, n_r$, and integers $m_1, p_1, \ldots, m_r, p_r$ with $0 \le m_i, p_i < n$ and*

$(m_i, p_{i+1} - p_i) \neq (0,0)$ *for* $1 \leq i \leq r$, *we have*

$$\tau((\sigma(v^{p_1})\sigma(g_1^{n_1})\sigma(v^{p_1})^*)\,\sigma(u^{m_1})\,(\sigma(v^{p_2})\sigma(g_1^{n_2})\sigma(v^{p_2})^*)\cdots\left(\sigma(v^{p_r})\sigma(g_1^{n_r})\sigma(v^{p_r^*})\right)\sigma(u^{m_r}))=0,$$

The last equality is equivalent to:

$$\tau(\sigma(g_{1+p_1}^{n_1})\sigma(u^{m_1})\sigma(g_{1+p_2}^{n_2})\cdots\sigma(g_{1+p_r}^{n_r})\sigma(u^{m_r}))=0.$$

On the other hand, by the freeness of u and $g_1, \ldots, g_n$ in $\mathbb{Z}_n * F_n$, we know, if $(m_i, p_{i+1} - p_i) \neq (0,0)$ for $1 \leq i \leq r$ then

$$g_{1+p_1}^{n_1} u^{m_1} g_{1+p_2}^{n_2} \cdots g_{1+p_r}^{n_r} u^{m_r} \quad \text{ is a reduced word in } \mathbb{Z}_n * F_n.$$

Therefore

$$\tau(\sigma(g_{1+p_1}^{n_1})\sigma(u^{m_1})\sigma(g_{1+p_2}^{n_2})\cdots\sigma(g_{1+p_r}^{n_r})\sigma(u^{m_r}))=0.$$

This implies that $\{\sigma(v), \sigma(u)\}$ and $\{\sigma(g_1)\}$ are free with respect to τ in $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$. This ends the proof of the claim.

[Continue the proof of the lemma:] Since u and v are two natural generators of the group $\mathbb{Z}_n$ and $\sigma(u)\sigma(v) = \gamma\sigma(v)\sigma(u)$ where γ is the n-th root of the unit, the C*-subalgebra $\mathcal{B}$ generated by $\sigma(u)$ and $\sigma(v)$ in $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$ is $*$-isomorphic to $\mathcal{M}_n(\mathbb{C})$. By Claim, we know that $\mathcal{B}$ and $\sigma(g_1)\mathcal{B}\sigma(g_1)^*$ are free with respect to τ in $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$. Since τ is a faithful tracial state on $C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n$, τ is also a faithful tracial state on the C*-subalgebra generated by $\mathcal{B}$ and $\sigma(g_1)\mathcal{B}\sigma(g_1)^*$. Combining with the fact that $\mathcal{B}$ is $*$-isomorphic to $\mathcal{M}_n(\mathbb{C})$, we know that

$$(\mathcal{M}_n(\mathbb{C}), \tau_n) *_{red} (\mathcal{M}_n(\mathbb{C}), \tau_n) \simeq (\mathcal{B}, \tau|_{\mathcal{B}}) *_{red} (\mathcal{B}, \tau|_{\mathcal{B}}) \simeq C^*(\mathcal{B}, \sigma(g_1)\mathcal{B}\sigma(g_1)^*) \subseteq C_r^*(\mathbb{Z}_n * F_n) \rtimes_{\alpha,r} \mathbb{Z}_n.$$

By Lemma 3.2.2 and Corollary 3.2.1, we know that

$$(\mathcal{M}_n(\mathbb{C}), \tau_n) *_{red} (\mathcal{M}_n(\mathbb{C}), \tau_n)$$

is an MF algebra. □

Definition 3.2.1. *Suppose that*

$$\mathcal{B} \simeq \mathcal{M}_{n_1}(\mathbb{C}) \oplus \mathcal{M}_{n_2}(\mathbb{C}) \oplus \cdots \oplus \mathcal{M}_{n_r}(\mathbb{C})$$

is a finite dimensional C-algebra. Let* τ_{n_i} *be the normalized tracial state on* $\mathcal{M}_{n_i}(\mathbb{C})$ *for each* $1 \leq i \leq r$. *Moreover every element* x *in* $\mathcal{B}$ *can be written as*

$$x = x_1 \oplus x_2 \oplus \cdots \oplus x_r, \qquad \text{with each } x_i \in \mathcal{M}_{n_i}(\mathbb{C}), \ \forall\ 1 \leq i \leq r.$$

Then a tracial state τ *on* $\mathcal{B}$ *is called a rational tracial state if there are rational numbers* $0 \leq \alpha_1, \ldots, \alpha_r \leq 1$ *such that*

$$\tau(x) = \alpha_1\tau_{n_1}(x_1) + \cdots + \alpha_r\tau_{n_r}(x_r), \qquad \forall\ x \in \mathcal{B}.$$

Proposition 3.2.1. *Suppose that* $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *are finite dimensional C*-algebras with faithful rational tracial states* τ_1, *and* τ_2 *respectively. Then*

$$(\mathcal{B}_1, \tau_1) *_{red} (\mathcal{B}_2, \tau_2)$$

is an MF algebra.

Proof. Since both τ_1 and τ_2 are faithful rational tracial states on $\mathcal{B}_1$, and $\mathcal{B}_2$ respectively, there are a positive integer n and trace-preserving, faithful, unital $*$-monomorphisms $\pi_1 : \mathcal{B}_1 \to \mathcal{M}_n(\mathbb{C})$, and $\pi_2 : \mathcal{B}_2 \to \mathcal{M}_n(\mathbb{C})$, such that

$$\tau_n(\pi_1(x_1)) = \tau_1(x_1) \qquad \textit{and} \qquad \tau_n(\pi_2(x_2)) = \tau_2(x_2), \qquad \forall\ x_1 \in \mathcal{B}_1, x_2 \in \mathcal{B}_2,$$

where τ_n is the tracial state on $\mathcal{M}_n(\mathbb{C})$. Therefore,

$$(\mathcal{B}_1, \tau_1) *_{red} (\mathcal{B}_2, \tau_2) \subseteq (\mathcal{M}_n(\mathbb{C}), \tau_n) *_{red} (\mathcal{M}_n(\mathbb{C}), \tau_n).$$

By Lemma 3.2.2, we know that $(\mathcal{B}_1, \tau_1) *_{red} (\mathcal{B}_2, \tau_2)$ is an MF algebra. $\square$

3.3. Reduced free products of unital AH algebras. Recall the definition of unital AF algebra as follows.

Definition 3.3.1. *A unital separable C*-algebra $\mathcal{A}$ is called approximately finite dimensional (AF) algebra if for every $x_1, \ldots, x_n$ in $\mathcal{A}$ and every $\epsilon > 0$, there is a finite dimensional C*-subalgebra $I_{\mathcal{A}} \in \mathcal{B} \subseteq \mathcal{A}$ satisfying*

$$\max_{1 \le i \le n} dist(x_i, \mathcal{B}) \le \epsilon.$$

The following lemma is quite useful.

Lemma 3.3.1. *Suppose that $\mathcal{A}$ is a separable C*-algebra. Assume for every $x_1, \ldots, x_n$ in $\mathcal{A}$ and every $\epsilon > 0$, there is an MF algebra $\mathcal{A}_1$ such that*

$$\{x_1, \ldots, x_n\} \subseteq_\epsilon \mathcal{A}_1 \qquad \textit{(in the sense of Definition 3.3.1).}$$

Then $\mathcal{A}$ is also an MF algebra.

Proof. Assume that $x_1, \ldots, x_n$ is a family of self-adjoint elements in $\mathcal{A}$ and $\epsilon > 0$. Suppose that $P_1, \ldots, P_r$ is a family of noncommutative polynomials in $\mathbb{C}\langle X_1, \ldots, X_n\rangle$. Therefore, by condition on $\mathcal{A}$, we know that there is an MF algebra $\mathcal{A}_1$ and a family of self-adjoint elements $y_1, \ldots, y_n$ in $\mathcal{A}_1$ such that, for all $1 \le j \le r$

$$|\|P_j(x_1, \ldots, x_n)\| - \|P_j(y_1, \ldots, y_n)\|| \le \epsilon/2.$$

Since $\mathcal{A}_1$ is an MF algebra, there are a positive integer k and self-adjoint matrices

$$A_1, \ldots, A_n \in \mathcal{M}_k^{s.a.}(\mathbb{C})$$

such that, for all $1 \le j \le r$,

$$|\|P_j(A_1, \ldots, A_n)\| - \|P_j(y_1, \ldots, y_n)\|| \le \epsilon/2.$$

It follows that, for all $1 \le j \le r$,

$$|\|P_j(A_1, \ldots, A_n)\| - \|P_j(x_1, \ldots, x_n)\|| \le \epsilon.$$

By Lemma 2.4.1, we know that $\mathcal{A}$ is an MF algebra. $\square$

Lemma 3.3.2. *Suppose that $\mathcal{B}_1$ and $\mathcal{B}_2$ are finite dimensional C^*-algebras with faithful tracial states τ, and ψ respectively. Then*

$$(\mathcal{B}_1, \tau) *_{red} (\mathcal{B}_2, \psi)$$

is an MF algebra.

Proof. Suppose that $x_1 \ldots, x_n$ is a family of elements in $(\mathcal{B}_1, \tau) *_{red} (\mathcal{B}_2, \psi)$ and $\epsilon > 0$ is a positive number.

Apparently, there is a sequence of faithful rational tracial states τ_α, $\alpha = 1, 2 \ldots$, on $\mathcal{B}_1$ such that

$$\lim_{\alpha \to \infty} \tau_\alpha(b) = \tau(b), \qquad \forall\ b \in \mathcal{B}_1.$$

Thus by Lemma 3.1.3, there is a positive integer α such that

$$\{x_1, \ldots, x_n\} \subseteq_\epsilon (\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi), \quad \text{(in the sense of Definition 3.3.1)}$$

whence there are $y_1, \ldots, y_n$ in $(\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi)$ and unital faithful $*$-representations ρ_1 of $(\mathcal{B}_1, \tau) *_{red} (\mathcal{B}_2, \psi)$ and ρ_2 of $(\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi)$ on a Hilbert space $\mathcal{H}$ such that

$$\max_{1 \le i \le n} \|\rho_1(x_i) - \rho_2(y_i)\| \le \epsilon.$$

Applying Lemma 3.1.3 again, we know that there is a faithful rational tracial state ψ_β on $\mathcal{B}_2$ such that

$$\{y_1, \ldots, y_n\} \subseteq_\epsilon (\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi_\beta).$$

I.e. there are $z_1, \ldots, z_n$ in $(\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi_\beta)$ and unital faithful $*$-representations ρ_3 of $(\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi)$ and ρ_4 of $(\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi_\beta)$ on a Hilbert space $\mathcal{K}$ such that

$$\max_{1 \le i \le n} \|\rho_3(y_i) - \rho_4(z_i)\| \le \epsilon.$$

Without loss of generality, we can assume that both ρ_2 and ρ_3 are unital, faithful, essential representations, i.e. there is no nonzero compact operator in the ranges of ρ_2 and ρ_3. By a result in [**25**], there is a sequence of unitaries $u_k : \mathcal{H} \to \mathcal{K}$, for $k = 1, 2, \ldots$, such that

$$\limsup_{k \to \infty} \|\rho_2(y) - u_k^* \rho_3(y) u_k\| = 0, \qquad \forall\ y \in (\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi).$$

It follows that, $\forall\ 1 \le i \le n$,

$$\begin{aligned}
&\limsup_{k \to \infty} \|\rho_1(x_i) - u_k^* \rho_4(z_i) u_k\| \\
&\qquad \le \limsup_{k \to \infty} \left(\|\rho_1(x_i) - \rho_2(y_i)\| + \|\rho_2(y_i) - u_k^* \rho_3(y_i) u_k\| + \|u_k^* \rho_3(y_i) u_k - u_k^* \rho_4(z_i) u_k\| \right) \\
&\qquad \le 2\epsilon.
\end{aligned}$$

Altogether, we have that

$$\{x_1, \ldots, x_n\} \subseteq_{3\epsilon} (\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi_\beta).$$

By Proposition 3.2.1, we know that $(\mathcal{B}_1, \tau_\alpha) *_{red} (\mathcal{B}_2, \psi_\beta)$ is an MF algebra. Thus by Lemma 3.3.1, we know that $(\mathcal{B}_1, \tau) *_{red} (\mathcal{B}_2, \psi)$ is an MF algebra. $\square$

Theorem 3.3.1. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital separable AF subalgebras with faithful tracial states τ_1, and τ_2 respectively. Then*

$$(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$$

is an MF algebra.

Proof. Suppose that $x_1 \ldots, x_n$ is a family of elements in $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$ and $\epsilon > 0$ is a positive number.

By the definition of AF algebra, we know that there are finite dimensional C*-algebras $I_{\mathcal{A}_i} \in \mathcal{B}_i \subseteq \mathcal{A}_i$ for $i = 1, 2$ such that

$$\{x_1, \ldots, x_n\} \subseteq_\epsilon \text{the C}^*\text{-subalgebra generated by } \mathcal{B}_1 \text{ and } \mathcal{B}_2 \text{ in } (\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2) \ .$$

Since $\tau_1 * \tau_2$ is a faithful tracial state of $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$ and $I_{\mathcal{A}_1} \in \mathcal{B}_1$, $I_{\mathcal{A}_2} \in \mathcal{B}_2$, we know that the C*-subalgebra generated by $\mathcal{B}_1$ and $\mathcal{B}_2$ in $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$ is $*$-isomorphic to

$$(\mathcal{B}_1, \tau_1|_{\mathcal{B}_1}) *_{red} (\mathcal{B}_2, \tau_2|_{\mathcal{B}_2}).$$

Therefore,

$$\{x_1, \ldots, x_n\} \subseteq_\epsilon (\mathcal{B}_1, \tau_1|_{\mathcal{B}_1}) *_{red} (\mathcal{B}_2, \tau_2|_{\mathcal{B}_2}).$$

By Lemma 3.3.2, we know that $(\mathcal{B}_1, \tau_1|_{\mathcal{B}_1}) *_{red} (\mathcal{B}_2, \tau_2|_{\mathcal{B}_2})$ is an MF algebra. It follows from Lemma 3.3.1 that

$$(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$$

is an MF algebra. □

Lemma 3.3.3. *The following statements are true:*

(1) *Suppose that X is a compact metric space and $C(X)$ is the unital C*-algebra consisting all continuous functions on X. Let τ be a faithful tracial state on $C(X)$. Then there are a unital separable AF algebra $\mathcal{A}$ with a faithful trace ψ and a unital embedding $\rho : C(X) \to \mathcal{A}$ such that $\tau(x) = \psi(\rho(x))$ for all $x \in C(X)$.*

(2) *Suppose that $\mathcal{B} \simeq \oplus_{i=1}^k \big(\mathcal{M}_{n_i}(\mathbb{C}) \otimes C(X_i)\big)$ is a unital separable C*-algebra with a faithful tracial state τ, where each X_i is a compact metric space for $1 \le i \le k$. Then there are a unital separable AF algebra $\mathcal{A}$ with a faithful trace ψ and a unital embedding $\rho : \mathcal{B} \to \mathcal{A}$ such that $\tau(x) = \psi(\rho(x))$ for all $x \in \mathcal{B}$.*

Proof. It suffices to prove (1). Since X is a compact metric space, $C(X)$ is a separable C*-algebra. We might assume that $\{x_n\}_{n=1}^\infty$ is a dense subset in $C(X)$. Let ρ be the GNS representation of $C(X)$ on the Hilbert space $L^2(C(X), \tau)$. Any element a in $C(X)$ corresponds to a vector $\hat{a}$ in $L^2(C(X), \tau)$. Let ψ be the vector state defined by $\psi(T) = \langle T\hat{1}, \hat{1}\rangle$ for all T in $B(L^2(C(X), \tau))$, where 1 is the unit of $C(X)$. Let $\mathcal{M}$ be the von Neumann algebra generated by $\rho(C(X))$ in $B(L^2(C(X), \tau))$. Since τ is a faithful trace of $C(X)$ and ρ is the GNS representation of the unital C*-algebra $C(X)$, we know ρ is a faithful $*$-representation of $C(X)$ and $\mathcal{M}$ is an abelian von Neumann algebra with a faithful tracial state ψ.

By spectral theory, for each x_n in $C(X)$, there is a sequence of projections $\{p_{n,k}\}_{k=1}^\infty$ in $\mathcal{M}$ such that $\rho(a)$ is in the C*-subalgebra, $C^*(\{p_{n,k}\}_{k=1}^\infty)$, generated by $\{p_{n,k}\}_{k=1}^\infty$ in $\mathcal{M}$. Let $\mathcal{A}$

be the unital C*-algebra generated by $\{p_{n,k}\}_{n,k=1}^{\infty}$ in $\mathcal{M}$. Therefore $\mathcal{A}$ is an unital AF algebra. Moreover ψ is a faithful tracial state on $\mathcal{A}$ and ρ is a unital embedding from $C(X)$ into $\mathcal{A}$ satisfying $\tau(x) = \psi(\rho(x))$ for all $x \in C(X)$. □

Recall the definition of AH algebra in the sense of Blackadar (see Definition 2.1 in [**4**]).

Definition 3.3.2. *A unital separable C*-algebra $\mathcal{A}$ is an approximately homogeneous (AH) C*-algebra if $\mathcal{A}$ is an inductive limit of a sequence of homogeneous C*-algebras $\mathcal{A}_m, m = 1, 2, \ldots,$ where each $\mathcal{A}_m = \oplus_{i=1}^{k_m}\big(\mathcal{M}_{[m,n_i]}(\mathbb{C}) \otimes C(X_{[m,i]})\big)$ and each $X_{[m,i]}$ is a compact matric space.*

Theorem 3.3.2. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital separable AH algebras with faithful tracial states τ_1, and τ_2 respectively. Then*

$$(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$$

is an MF algebra.

Proof. For each $i = 1, 2$, the unital AH algebra $\mathcal{A}_i$ is an inductive limit of homogeneous subalgebras $\{\mathcal{A}_m^{(i)}\}_{m=1}^{\infty}$, each of which is $*$-isomorphic to some $\oplus_{j=1}^{k}(\mathcal{M}_{n_j}(\mathbb{C}) \otimes C(X_j))$ where each X_j is a compact metric space for $1 \le j \le k$. By Lemma 3.3.3, we know for every $x_1, \ldots, x_n$ in $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$ and $\epsilon > 0$, there are a positive integer m, unital AF algebras $\mathcal{D}_1$ and $\mathcal{D}_2$ with faithful tracial states ψ_1, and ψ_2 respectively, such that

$$\{x_1, \ldots, x_n\} \subset_{\epsilon} (A_m^{(1)}, \tau_1) *_{red} (A_m^{(2)}, \tau_2) \subseteq (\mathcal{D}_1, \psi_1) *_{red} (\mathcal{D}_2, \psi_2).$$

By Lemma 3.3.1 and Theorem 3.3.1, we know that $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$ is an MF algebra. □

Recall a C*-algebra $\mathcal{A}$ is called a *local AH-algebra* if for every $\epsilon > 0$ and for every finite subset $a_1, \ldots, a_n$ of $\mathcal{A}$ there is a C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ which (i) is homogeneous, i.e. $\mathcal{B}$ is $*$-isomorphic to a C*-algebra of the form $\oplus_{i=1}^{k_m}\big(\mathcal{M}_{[m,n_i]}(\mathbb{C}) \otimes C(X_{[m,i]})\big)$, and (ii) contains elements $b_1, \ldots, b_n$ with $\|a_j - b_j\| \le \epsilon$ for $j = 1, \ldots, n$. Similar argument in the proof of Theorem 3.3.2 proves the following statement.

Proposition 3.3.1. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital, separable, local AH algebras with faithful tracial states τ_1, and τ_2 respectively. Then*

$$(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2) \qquad \text{is an MF algebra.}$$

Apparently, Theorem 3.3.2 can be generalized as follows.

Theorem 3.3.3. *Suppose that $\mathcal{A}_i$, $i = 1, \ldots, n$, is a family of unital separable AH algebras with faithful tracial states τ_i. Then*

$$(\mathcal{A}_1, \tau_1) *_{red} \cdots *_{red} (\mathcal{A}_n, \tau_n) \qquad \text{is an MF algebra.}$$

Proof. Let $\mathcal{A} = \mathcal{A}_1 \otimes_{min} \cdots \otimes_{min} \mathcal{A}_n$ be the minimal tensor product of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ and $\tau = \tau_1 \otimes_{min} \cdots \otimes_{min} \tau_n$ be the tensor product of the tracial states $\tau_1, \ldots, \tau_n$. Then $\mathcal{A}$ is an AH algebra with a faithful trace τ. Let $C_r^*(\mathbb{Z})$ be the reduced C*-algebra of the group $\mathbb{Z}$ with a canonical faithful trace $\tau_{\mathbb{Z}}$.

By Proposition 3.3.1 or Theorem 3.3.2, we know that $(\mathcal{A},\tau)*_{red}(C_r^*(\mathbb{Z}),\tau_{\mathbb{Z}})$ is an MF algebra. And we note that

$$(\mathcal{A}_1,\tau_1)*_{red}\cdots *_{red}(\mathcal{A}_n,\tau_n)\subseteq(\mathcal{A},\tau)*_{red}\cdots *_{red}(\mathcal{A},\tau)\subseteq(\mathcal{A},\tau)*_{red}(C_r^*(\mathbb{Z}),\tau_{\mathbb{Z}}).$$

Therefore

$$(\mathcal{A}_1,\tau_1)*_{red}\cdots *_{red}(\mathcal{A}_n,\tau_n)$$

is an MF algebra. □

COROLLARY 3.3.1. *Suppose that $(\mathcal{A},\tau)$ is a C*-free probability space. Let $x_1,\ldots,x_n$ be a family of self-adjoint elements in $\mathcal{A}$ such that $x_1,\ldots,x_n$ are free with respect to τ. Then the C*-subalgebra generated by $x_1,\ldots,x_n$ in $\mathcal{A}$ is an MF algebra. In particular,*

$$\delta_{top}(x_1,\ldots,x_n)\geq 0,$$

where $\delta_{top}(x_1,\ldots,x_n)$ is the Voiculescu's topological free entropy dimension.

PROOF. It follows directly from the definition of Voiculescu's topological free entropy dimension [**30**] and the definition of MF algebra (see Lemma 2.4.1). □

More discussions on topological free entropy dimension can be found in [**19**], [**20**], and [**21**].

3.4. Reduced free product of unital ASH algebras. Recall if $\mathcal{B}$ is a unital sub-homogeneous C*-algebra (or equivalently, a C*-subalgebra of a homogeneous C*-algebra), then all irreducible $*$-representations of $\mathcal{B}$ are finite dimensional with $dim\leq n$ for some positive integer $n\in\mathbb{N}$. Suppose τ is a faithful tracial state of $\mathcal{B}$ and $\mathcal{H}$ is the Hilbert space $L^2(\mathcal{B},\tau)$. Each element a in $\mathcal{B}$ corresponds to a vector $\hat{a}$ in $L^2(\mathcal{B},\tau)$. Let π be the GNS representation of $\mathcal{B}$ on the Hilbert space $L^2(\mathcal{B},\tau)$. Then the von Neumann algebra, $\pi(\mathcal{B})''$, generated by $\pi(\mathcal{B})$ in $B(L^2(\mathcal{B},\tau))$ has the form:

$$\pi(\mathcal{B})''\simeq\oplus_{k=1}^n(\mathcal{M}_k(\mathbb{C})\otimes\mathcal{B}_k),$$

where each $\mathcal{B}_k$ is either a unital abelian von Neumann algebra or 0. Let ψ be the vector state defined by $\psi(T)=\langle T\hat{1},\hat{1}\rangle$ for all T in $B(L^2(\mathcal{B},\tau))$, where 1 is the unit of $\mathcal{B}$. Since τ is a faithful trace of $\mathcal{B}$ and π is the GNS representation of the unital C*-algebra $\mathcal{B}$, we know ψ is a faithful trace of $\pi(\mathcal{B})''$ and

$$\tau(x)=\psi(\pi(x)),\qquad\forall\ x\in\mathcal{B}.$$

Hence π is a unital, trace-preserving, embedding from $\mathcal{B}$ into $\pi(\mathcal{B})''$ such that

$$(\mathcal{B},\tau)\subseteq(\pi(\mathcal{B})'',\psi).$$

By similar argument as in Lemma 3.3.3, we have the following result.

LEMMA 3.4.1. *Suppose that $\mathcal{B}$ is a unital separable sub-homogeneous C*-algebra with a faithful tracial state τ. Then there are a unital separable AF algebra $\mathcal{A}$ with a faithful trace ψ and a unital embedding $\rho:\mathcal{B}\to\mathcal{A}$ such that $\tau(x)=\psi(\rho(x))$ for all $x\in\mathcal{B}$.*

Recall an ASH algebra (approximately sub-homogeneous C*-algebra) is an inductive limit of a sequence of sub-homogeneous C*-algebras. By similar arguments as in Theorem 3.3.2 and Theorem 3.3.3 (using Lemma 3.4.1 instead of Lemma 3.3.3), we have the following result.

THEOREM 3.4.1. *Suppose that $\mathcal{A}_i$, $i = 1, \ldots, n$, is a family of unital separable ASH algebras with faithful tracial states τ_i. Then*

$$(\mathcal{A}_1, \tau_1) *_{red} \cdots *_{red} (\mathcal{A}_n, \tau_n)$$

is an MF algebra.

4. BDF Extension Semigroups and Reduced Free Products of AH Algebras

Recall the definition of quasidiagonal C*-algebra as follows.

DEFINITION 4.0.1. *A set of elements $\{a_1, \ldots, a_n\} \subseteq B(\mathcal{H})$ is quasidiagonal if there is an increasing sequence of finite-rank projections $\{p_i\}_{i=1}^{\infty}$ on $\mathcal{H}$ tending strongly to the identity such that $\|a_j p_i - p_i a_j\| \to 0$ as $i \to \infty$ for any $1 \le j \le n$. A separable C*-algebra $\mathcal{A} \subseteq B(\mathcal{H})$ is quasidiagonal if there is an increasing sequence of finite-rank projections $\{p_i\}_{i=1}^{\infty}$ on $\mathcal{H}$ tending strongly to the identity such that $\|xp_i - p_i x\| \to 0$ as $i \to \infty$ for any $x \in \mathcal{A}$. An abstract separable C*-algebra $\mathcal{A}$ is quasidiagonal if there is a faithful $*$-representation $\pi : \mathcal{A} \to B(\mathcal{H})$ such that $\pi(\mathcal{A}) \subseteq B(\mathcal{H})$ is quasidiagonal.*

By a result of Rosenberg, we know that $C_r^*(F_2)$ is not quasidiagonal.

We will use the fact that a C*-subalgebra of a separable quasidiagonal C*-algebra is also quasidiagonal. In other words, a separable C*-algebra, containing a non-quasidiagonal C*-subalgebra, is not quasidiagonal.

4.1. Non-quasidiagonality of reduced free products of AH algebras. In this subsection, we are going to discuss quasidiagonality of reduced free products of unital C*-algebras. Some of the conclusions stated in this subsection are direct consequences of results from other literature.

The following result might have been known to experts. For the purpose of completeness, we include it here.

THEOREM 4.1.1. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital separable C*-algebras with faithful tracial states τ_1, and τ_2 respectively. If $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy Avitzour's condition,* i.e. there are unitaries $u \in \mathcal{A}_1$ and $v, w \in \mathcal{A}_2$ such that

$$\tau_1(u) = \tau_2(v) = \tau_2(w) = \tau_2(w^*v) = 0,$$

then

$$(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$$

is not a quasidiagonal C-algebra.*

PROOF. Let $a = uvuv$ and $b = uwuw$ be unitaries in $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$. Then we know that a and b are two Haar unitary elements in $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$ with respect to the trace $\tau_1 * \tau_2$. We note that

$$\begin{array}{llll} ab = uvuvuwuw & ab^* = uvu(vw^*)u^*w^*u^* & a^*b = v^*u^*(v^*w)uw & a^*b^* = v^*u^*v^*u^*w^*u^*w^*u^* \\ ba = uwuwuvuv & ba^* = uwu(wv^*)u^*v^*u^* & b^*a = w^*u^*(w^*v)uv & b^*a^* = w^*u^*w^*u^*v^*u^*v^*u^* \end{array}$$

Now it is not hard to check that a and b are free with respect to $\tau_1 * \tau_2$. In other words, $C_r^*(F_2)$ is a C*-subalgebra of $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$. Since $C_r^*(F_2)$ is not a quasidiagonal C*-algebra, $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$ is not a quasidiagonal C*-algebra. □

The following result of N. Brown (see Corollary 4.3.6 in [**10**]) is also useful in determining the quasidiagonality of a unital C*-algebra.

LEMMA 4.1.1 (Brown). *Suppose that $\mathcal{A}$ is a unital, separable, exact C*-algebra with a unique trace τ. Let $\rho : \mathcal{A} \to B(L^2(\mathcal{A}, \tau))$ be the GNS representation of $\mathcal{A}$ on the Hilbert space $L^2(\mathcal{A}, \tau)$. If $\mathcal{A}$ is quasidiagonal, then $\rho(\mathcal{A})''$, the von Neumann algebra generated by $\rho(\mathcal{A})$ in $B(L^2(\mathcal{A}, \tau))$, is a hyperfinite von Neumann algebra.*

PROPOSITION 4.1.1. *Suppose that $C(\mathbb{T})$ is the unital C*-algebra consisting all continuous functions on the unit circle $\mathbb{T}$ and τ is a faithful trace of $C(\mathbb{T})$ induced by the Lesbeague measure on $\mathbb{T}$. Suppose that $\mathcal{B} \neq \mathbb{C}$ is a unital, separable, C*-algebra with a faithful tracial state ψ. Then*

$$(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi)$$

is not a quasidiagonal C-algebra.*

PROOF. We might assume that $\mathcal{B}$ is an exact C*-algebra. In fact, let $1 \neq v$ be a unitary in $\mathcal{B}$ and $I_{\mathcal{B}} \in \mathcal{B}_1$ be a unital C*-subalgebra of $\mathcal{B}$ generated by v in $\mathcal{B}$. Since $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}_1, \psi)$ is a C*-subalgebra of $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi)$, to show $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi)$ is not quasidiagonal it suffices to show that $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}_1, \psi)$ is not quasidiagonal. Apparently $\mathcal{B}_1 \neq \mathbb{C}$ is a unital exact C*-algebra with a faithful trace ψ.

By Dykema's result in Theorem 2 of [**13**], we know that $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi)$ is a simple C*-algebra with a unique tracial state $\tau * \psi$. Also by his result in Theorem 3.5 of [**14**], we know that $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi)$ is an exact C*-algebra.

Let ρ be the GNS representation of $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi)$ on the Hilbert space $L^2((C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi), \tau * \psi)$. Assume that $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi)$ is a quasidiagonal C*-algebra. Then by Lemma 4.1.1, we know that $\rho((C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi))''$ is a hyperfinite von Neumann algebra. Let u be a Haar unitary in $C(\mathbb{T})$ with respect to τ and $v \neq 1$ be a unitary in $\mathcal{B}$. Then by Voiculescu's result in [**28**], we know that

$$\delta_0(\rho(u), \rho(v)) = \delta_0(\rho(u)) + \delta_0(\rho(v)) = 1 + \delta_0(\rho(v)) > 1,$$

where δ_0 is the modified free entropy dimension for finite von Neumann algebras. On the other hand, since $\rho((C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi))''$ is a hyperfinite von Neumann algebra, by [**29**] or [**18**], we know

$$\delta_0(\rho(u), \rho(v)) \leq 1.$$

This is the contradiction. Hence $\rho((C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi))''$ is not a hyperfinite von Neumann algebra. It follows that $(C(\mathbb{T}), \tau) *_{red} (\mathcal{B}, \psi)$ is not a quasidiagonal C*-algebra. □

The following useful result was obtained by Dykema in Proposition 2.8 of [**13**].

LEMMA 4.1.2 (Dykema). *Let $\mathcal{A} = \mathcal{A}_1 \oplus \mathcal{A}_2$ be a direct sum of unital C*-algebras. Write $p = 1 \oplus 0 \in \mathcal{A}$ and let $\phi_{\mathcal{A}}$ be a state on A, such that $0 < \alpha = \phi_{\mathcal{A}}(p) < 1$. Let $\mathcal{B}$ be a unital C*-algebra with a state $\phi_{\mathcal{B}}$ and let*

$$(\mathcal{D}, \phi) = (\mathcal{A}, \phi_{\mathcal{A}}) *_{red} (\mathcal{B}, \phi_{\mathcal{B}})$$

Let $\mathcal{D}_1$ be the C-subalgebra of $\mathcal{D}$ generated by $\mathbb{C}p + (0 \oplus \mathcal{A}_2) \subseteq \mathcal{A}$ together with $\mathcal{B}$. Then $p\mathcal{D}p$ is generated by $p\mathcal{D}_1 p$ and $\mathcal{A}_1 \oplus 0 \subseteq \mathcal{A}$, which are free in $(p\mathcal{D}p, \frac{1}{\alpha}\phi|_{p\mathcal{D}p})$, i.e.*

$$(p\mathcal{D}_1 p, \frac{1}{\alpha}\phi|_{p\mathcal{D}_1 p}) *_{red} (\mathcal{A}_1, \frac{1}{\alpha}\phi_{\mathcal{A}}|_{\mathcal{A}_1}) \simeq (p\mathcal{D}p, \frac{1}{\alpha}\phi|_{p\mathcal{D}p}) \subseteq \mathcal{D}.$$

PROPOSITION 4.1.2. *Let $C(\mathbb{T})$ be the unital C*-algebra consisting all continuous functions on the unit circle $\mathbb{T}$ and τ a faithful trace of $C(\mathbb{T})$ induced by the Lesbeague measure on $\mathbb{T}$. Let $\mathcal{A}_2$ and $\mathcal{B} \neq \mathbb{C}$ be unital separable C*-algebras with faithful traces τ_2, and ψ respectively. Let $\mathcal{A} = C(\mathbb{T}) \oplus \mathcal{A}_2$ with a faithful trace ϕ given by $\phi = \alpha\tau + (1-\alpha)\tau_2$ for some $0 < \alpha < 1$. Then*

$$(\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)$$

is not a quasidiagonal C-algebra.*

PROOF. Let $(\mathcal{D}, \phi * \psi) \simeq (\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)$. By Lemma 4.1.2, there is a unital C*-subalgebra $\mathcal{D}_2 \neq \mathbb{C}$ in $\mathcal{D}$, such that $(C(\mathbb{T}), \tau) *_{red} (\mathcal{D}_2, \frac{1}{(\phi*\psi)(I_{\mathcal{D}_2})}(\phi*\psi)|_{\mathcal{D}_2})$ can be embedded (not necessary to be unital) into $(\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)$. Combining with Proposition 4.1.1, we completed the proof. $\square$

Recall a unital C*-algebra $\mathcal{A}$ with a faithful trace ϕ is diffuse if there is a unitary u such that $\phi(u^n) = 0$ for all $n \neq 0$, i.e. u is a Haar unitary in $\mathcal{A}$.

DEFINITION 4.1.1. *Suppose that $\mathcal{A}$ is a unital C*-algebra with a faithful tracial state ϕ. Then $(\mathcal{A}, \phi)$ is called* partially diffuse *if there is a partial isometry v in $\mathcal{A}$ such that $vv^* = v^*v$ and $\phi(v^n) = 0$ for all $n \neq 0$.*

THEOREM 4.1.2. *Suppose that $\mathcal{A}$ is a unital C*-algebra with a faithful tracial state ϕ. Then the following are equivalent:*

1. *$(\mathcal{A}, \phi)$ is partially diffuse;*
2. *There is a unital C*-subalgebra $\mathcal{B}$ of $\mathcal{A}$ such that $(\mathcal{B}, \frac{1}{\phi(I_{\mathcal{B}})}\phi|_{\mathcal{B}})$ is diffuse. (Note we don't require that $\mathcal{B}$ contains the unit of $\mathcal{A}$.)*
3. *There is a unital C*-subalgebra $\mathcal{C}$ of $\mathcal{A}$ such that*

   $$(\mathcal{C}, \frac{1}{\phi(I_{\mathcal{C}})}\phi) \simeq (C(\mathbb{T}), \tau),$$

   where $C(\mathbb{T})$ is the unital C-algebra consisting all continuous functions on the unit circle $\mathbb{T}$ and τ is a faithful trace of $C(\mathbb{T})$ induced by the Lesbeague measure on $\mathbb{T}$.*
4. *There is a self-adjoint element x in $\mathcal{A}$ satisfying:*

   Suppose that X is the spectrum of x in $\mathcal{A}$ and μ is the Borel measure on X induced from the trace ϕ. Then there are real numbers $a < b$ in X such that (i) $\mu|_{X \cap [a,b]}$ has no atom; (ii) the distance between $X \cap [a,b]$ and $X \setminus [a,b]$ is larger than 0.

PROOF. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is obvious. (1) $\Leftrightarrow$ (4) is by Lemma 4.2 in [**15**]. $\square$

PROPOSITION 4.1.3. *Let $\mathcal{A}$ and $\mathcal{B} \neq \mathbb{C}$ be unital separable C*-algebras with faithful traces ϕ, and ψ respectively. If $(\mathcal{A}, \phi)$ is partially diffuse, then*

$$(\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)$$

is not a quasidiagonal C-algebra.*

PROOF. Note that $\mathcal{A}$ is partially diffuse. By Theorem 4.1.2, there is a unital C*-subalgebra $\mathcal{C}$ of $\mathcal{A}$ such that

$$(\mathcal{C}, \frac{1}{\phi(I_{\mathcal{C}})}\phi) \simeq (C(\mathbb{T}), \tau),$$

where $C(\mathbb{T})$ is the unital C*-algebra consisting all continuous functions on the unit circle $\mathbb{T}$ and τ is a faithful trace of $C(\mathbb{T})$ induced by the Lesbeague measure on $\mathbb{T}$. Let $p = I_{\mathcal{C}}$ and $q = I_{\mathcal{A}} - p$ be the projections in $\mathcal{A}$. Then ϕ is a faithful trace on the unital C*-subalgebra $\mathcal{C}p + \mathbb{C}q$ of $\mathcal{A}$ and

$$(\mathcal{C}p + \mathbb{C}q, \phi) *_{red} (\mathcal{B}, \psi) \subseteq (\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi).$$

By Proposition 4.1.2, we know that

$$(\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)$$

is not a quasidiagonal C*-algebra. $\square$

LEMMA 4.1.3. *Let $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$ and $\mathcal{B} = \mathbb{C} \oplus \mathbb{C}$ with faithful traces ϕ, and ψ respectively. Let $p = 1 \oplus 0$ be a projection in $\mathcal{B}$. Then*

$$(p((\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi))p, \frac{1}{\psi(p)}(\phi * \psi)|_{p((\mathcal{A},\phi)*_{red}(\mathcal{B},\psi))p})$$

is partially diffuse.

PROOF. The C*-algebra $(\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)$ was totally determined in Theorem 13 of [**2**] (see also Proposition 2.7 in [**13**]). Thus the structure of $p(\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)p$ is also determined as listed in Theorem 13 of [**2**]. Now the rest follows from Lemma 4.2 in [**15**] (see also the proof of Lemma 4.1 in [**13**]). $\square$

LEMMA 4.1.4. *Let τ_1, τ_2 and ψ be faithful traces on the C* algebras $\mathcal{A}_1 = \mathbb{C} \oplus \mathbb{C}$, $\mathcal{A}_2 = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ and $\mathcal{A}_3 = \mathcal{M}_2(\mathbb{C})$ respectively. Then*

(i) $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2) = (\mathbb{C} \oplus \mathbb{C}, \tau_1) *_{red} (\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}, \tau_2)$ *is not a quasidiagonal C*-algebra;*
(ii) $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_3, \psi) = (\mathbb{C} \oplus \mathbb{C}, \tau_1) *_{red} (\mathcal{M}_2(\mathbb{C}), \psi)$ *is not a quasidiagonal C*-algebra.*

PROOF. (i) Let $\mathcal{B} = \mathbb{C} \oplus \mathbb{C} \oplus 0 \subset \mathcal{A}_2$ be a C*-subalgebra of $\mathcal{A}_2$. Let $p = 1 \oplus 1 \oplus 0$ and $q = 0 \oplus 0 \oplus 1$ be projections in $\mathcal{A}_2$. Let $\mathcal{D}_1$ be the C*-subalgebra generated by $\mathcal{A}_1$ and $\mathbb{C}p + \mathbb{C}q$ in $(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)$. Then

$$(\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2) \supseteq (\mathcal{D}_1, \tau_1 * \tau_2|_{\mathcal{D}_1}) \simeq (\mathbb{C} \oplus \mathbb{C}, \tau_1) *_{red} (\mathbb{C}p + \mathbb{C}q, \tau_2);$$

and, by Lemma 4.1.2, we have

Fact 1: $p\mathcal{D}_1p$ and $\mathcal{B}\oplus 0$ are free in $p\left((\mathcal{A}_1,\tau_1) *_{red} (\mathcal{A}_2,\tau_2)\right)p$ with respect to $\frac{1}{\tau_2(p)}(\tau_1 * \tau_2)|_{p((\mathcal{A}_1,\tau_1)*_{red}(\mathcal{A}_2,\tau_2))p}$.

By Lemma 4.1.3, we know that $(p\mathcal{D}_1p, \frac{1}{\tau_2(p)}(\tau_1 * \tau_2)|_{p\mathcal{D}_1p})$ is partially diffuse. Note that $\mathcal{B}\neq \mathbb{C}$. Combining with Proposition 4.1.3 and Fact 1, we know that the C*-subalgebra generated $p\mathcal{D}_1p$ and $\mathcal{B}\oplus 0$ in $(\mathcal{A}_1,\tau_1) *_{red} (\mathcal{A}_2,\tau_2)$ is not quasidiagonal. Hence $(\mathcal{A}_1,\tau_1) *_{red} (\mathcal{A}_2,\tau_2)$ is not quasidiagonal.

(ii) Note $\mathcal{A}_1=\mathbb{C}\oplus\mathbb{C}$. Let

$$u_1=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \textit{and} \qquad u_2=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

be unitaries in $\mathcal{M}_2(\mathbb{C})$. Then $\mathcal{A}_1$, $u_1\mathcal{A}_1u_1^*$ and $u_2\mathcal{A}_1u_2^*$ are free in $(\mathbb{C}\oplus\mathbb{C},\tau_1) *_{red} (\mathcal{M}_2(\mathbb{C}),\psi)$. Let $\mathcal{B}$ be C*-subalgebra generated by $\mathcal{A}_1$ and $u_1\mathcal{A}_1u_1^*$ in $(\mathbb{C}\oplus\mathbb{C},\tau_1) *_{red} (\mathcal{M}_2(\mathbb{C}),\psi)$. Then

$$\mathcal{B}\simeq (\mathcal{A}_1,\tau_1) *_{red} (\mathcal{A}_1,\tau_1) = (\mathbb{C}\oplus\mathbb{C},\tau_1) *_{red} (\mathbb{C}\oplus\mathbb{C},\tau_1);$$

and

Fact 2: $\mathcal{B}$ and $u_2\mathcal{A}_1u_2^*$ are free in $(\mathbb{C}\oplus\mathbb{C},\tau_1) *_{red} (\mathcal{M}_2(\mathbb{C}),\psi)$.

By Lemma 4.1.3, $\mathcal{B}$ is partially diffuse. Combining with Proposition 4.1.3 and Fact 2, we know that the C*-subalgebra generated $\mathcal{B}$ and $u_2\mathcal{A}_1u_2^*$ in $(\mathbb{C}\oplus\mathbb{C},\tau_1) *_{red} (\mathcal{M}_2(\mathbb{C}),\psi)$ is not quasidiagonal. Hence $(\mathbb{C}\oplus\mathbb{C},\tau_1) *_{red} (\mathcal{M}_2(\mathbb{C}),\psi)$ is not quasidiagonal. $\square$

The following proposition follows directly from preceding lemma.

PROPOSITION 4.1.4. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital separable C*-algebras with faithful tracial states τ_1, and τ_2 respectively. If there are C*-subalgebras $I_{\mathcal{A}_i}\in\mathcal{B}_i\subseteq\mathcal{A}_i$ for $i=1,2$ such that (i) $\mathcal{B}_1\simeq\mathbb{C}\oplus\mathbb{C}$; and (ii) either $\mathcal{B}_2\simeq\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$ or $\mathcal{B}_2\simeq\mathcal{M}_2(\mathbb{C})$, then*

$$(\mathcal{A}_1,\tau_1) *_{red} (\mathcal{A}_2,\tau_2)$$

is not a quasidiagonal C-algebra.*

We are are ready to show the following statement.

THEOREM 4.1.3. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital separable AF algebras with faithful tracial states τ_1, and τ_2 respectively. If $dim_{\mathbb{C}}\mathcal{A}\geq 2$ and $dim_{\mathbb{C}}\mathcal{A}_2\geq 3$, then*

$$(\mathcal{A}_1,\tau_1) *_{red} (\mathcal{A}_2,\tau_2)$$

is not a quasidiagonal C-algebra.*

PROOF. Note that both $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital AF algebras. Since $dim_{\mathbb{C}}\mathcal{A}_1\geq 2$, there is a C*-subalgebra $I_{\mathcal{A}_1}\in\mathcal{B}_1$ of $\mathcal{A}$ such that $\mathcal{B}_1\simeq\mathbb{C}\oplus\mathbb{C}$. Since $dim_{\mathbb{C}}\mathcal{A}_2\geq 3$, there is a C*-subalgebra $I_{\mathcal{A}_2}\in\mathcal{B}_2$ of $\mathcal{A}_2$ such that either $\mathcal{B}_2\simeq\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$ or $\mathcal{B}_2\simeq\mathcal{M}_2(\mathbb{C})$. Now it follows from Proposition 4.1.4, we know that $(\mathcal{A}_1,\tau_1) *_{red} (\mathcal{A}_2,\tau_2)$ is not a quasidiagonal C*-algebra. $\square$

4.2. BDF extension semigroups of reduced free products of AH algebras. Suppose $\mathcal{A}$ is a separable unital C*-algebra. The invariant $Ext(\mathcal{A})$ was introduced by Brown, Douglas and Fillmore in [**8**]. $Ext(\mathcal{A})$ is the set of equivalence classes $[\pi]$ of unital $*$-monomorphisms $\pi : \mathcal{A} \to \mathcal{C}(\mathcal{H})$, where $\mathcal{C}(\mathcal{H}) = B(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra for a separable Hilbert space $\mathcal{H} = l^2(\mathbb{Z})$. The equivalence relation is defined as follows:

$$\pi_1 \sim \pi_2 \Leftrightarrow \exists u \in \mathcal{U}(B(\mathcal{H}))\ \textit{such that}\ \forall a \in \mathcal{A} : \pi_1(a) = \rho(u)\pi_2(a)\rho(u)^*,$$

where $\mathcal{U}(B(\mathcal{H}))$ is the unitary group of $B(\mathcal{H})$ and $\rho : B(\mathcal{A}) \to \mathcal{C}(\mathcal{H})$ is the quotient map. There is a natural semigroup structure on $Ext(\mathcal{A})$. By a result of Voiculescu, $Ext(\mathcal{A})$ always has a unit. By a result of Choi and Effros, $Ext(\mathcal{A})$ is a group for every separable unital nuclear C*-algebras $\mathcal{A}$. In [**17**], Haagerup and Thorbjørnsen solved a long standing open problem by showing that $Ext(C_r^*(F_2))$ is not a group.

In this subsection, we consider the BDF extension semigroups of reduced free products of some unital AH algebras. First we recall a useful fact, which can be found in [**9**], [**17**] and [**26**]. (See also Lemma 2.4 in [**22**])

Lemma 4.2.1. *Suppose that $\mathcal{A}$ is a unital separable MF algebras. If $\mathcal{A}$ is not quasidiagonal, then $Ext(\mathcal{A})$ is not a group.*

By Theorem 3.3.2 (or Theorem 3.4.1), Theorem 4.1.1 and Lemma 4.2.1, we have the following result.

Theorem 4.2.1. *Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are unital separable AH (or ASH) algebras with faithful tracial states τ_1, and τ_2 respectively. If $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfy Avitzour's condition,* i.e. there are unitaries $u \in \mathcal{A}_1$ and $v, w \in \mathcal{A}_2$ such that

$$\tau_1(u) = \tau_2(v) = \tau_2(w) = \tau_2(w^*v) = 0,$$

then

$$Ext\big((\mathcal{A}_1, \tau_1) *_{red} (\mathcal{A}_2, \tau_2)\big) \qquad \textit{is not a group.}$$

By Theorem 3.3.2 (or Theorem 3.4.1), Proposition 4.1.3 and Lemma 4.2.1, we have the following result.

Theorem 4.2.2. *Let $\mathcal{A}$ and $\mathcal{B} \neq \mathbb{C}$ be unital separable AH (or ASH) algebras with faithful traces ϕ, and ψ respectively. If $\mathcal{A}$ is partially diffuse in the sense of Definition 4.1.1, then*

$$Ext\big((\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)\big) \qquad \textit{is not a group.}$$

By Theorem 3.3.2 (or Theorem 3.4.1), Theorem 4.1.3 and Lemma 4.2.1, we have the following result.

Theorem 4.2.3. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital separable AF algebras with faithful tracial states ϕ, and ψ respectively. If $dim_{\mathbb{C}}\mathcal{A} \geq 2$ and $dim_{\mathbb{C}}\mathcal{B} \geq 3$, then*

$$Ext\big((\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)\big) \qquad \textit{is not a group.}$$

Example 4.2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be irrational C*-algebras, or UHF algebras, with faithful traces ϕ, and ψ respectively. Then $Ext\big((\mathcal{A}, \phi) *_{red} (\mathcal{B}, \psi)\big)$ is not a group.*

5. Reduced Free Products of Tensor Products of Unital C*-algebras

In this section, we will discuss some generalizations of the results we obtained in the previous sections. Most of the results obtained in this section are parallel to the ones in section 3 and their proofs are also similar. Thus we skip most of the proofs of the results in this section and sketched them only if necessary.

The following notation will be used in this section. Suppose that G is a countable discrete group. We will denote $C_r^*(G)$ the reduced group C*-algebra of G and τ_G the canonical tracial state of $C_r^*(G)$.

5.1. A class of MF algebras.

Definition 5.1.1. *Let $\mathcal{S}$ be the set of all these pairs $(\mathcal{A}, \phi)$ such that $\mathcal{A}$ is a separable unital C*-algebra and ψ is a faithful tracial state of $\mathcal{A}$ satisfying $(\mathcal{A}, \psi) *_{red} (C_r^*(F_n), \tau_{F_n})$ is an MF algebra for every integer $n \geq 1$.*

By Theorem 3.3.3, we have the following result.

Proposition 5.1.1. *Suppose that $\mathcal{A}$ is a unital separable AH algebra and ψ is a faithful trace of $\mathcal{A}$. Then*

$$(\mathcal{A}, \psi) \in \mathcal{S},$$

where $\mathcal{S}$ is defined in Definition 5.1.1.

5.2. Minimal tensor products of unital C*-algebras with faithful traces. In this subsection, we will recall the definition of minimal tensor product of two unital C*-algebras when both C*-algebras have faithful traces.

Suppose that $\mathcal{A}_i$, $i = 1, 2$, are unital C*-algebras with faithful traces ψ_i. Each element a_i in $\mathcal{A}_i$ corresponds to a vector $\hat{a}_i$ in $\mathcal{H}_i = L^2(\mathcal{A}_i, \psi_i)$. Let

$$\rho_i : \mathcal{A}_i \to B(\mathcal{H}_i) = B(L^2(\mathcal{A}_i, \psi_i))$$

be the GNS representation of $\mathcal{A}_i$ such that

$$\psi_i(a_i) = \langle \rho_i(a_i)\hat{I}_{\mathcal{A}_i}, \hat{I}_{\mathcal{A}_i} \rangle, \qquad \forall\, a_i \in \mathcal{A}_i.$$

Then the C*-subalgebra generated by

$$\{\rho_1(a_1) \otimes I_{\mathcal{H}_2}, I_{\mathcal{H}_1} \otimes \rho_2(a_2) \mid a_i \in \mathcal{A}_i, i = 1, 2\}$$

in $B(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is the minimal tensor product of $\mathcal{A}$ and $\mathcal{B}$, and is denoted by $\mathcal{A} \otimes_{min} \mathcal{B}$.

Moreover, there is a canonical vector state $\psi = \psi_1 \otimes_{min} \psi_2$ defined on $\mathcal{A} \otimes_{min} \mathcal{B}$ as follows:

$$\psi(T) = \langle T(\hat{I}_{\mathcal{A}_1} \otimes \hat{I}_{\mathcal{A}_2}), \hat{I}_{\mathcal{A}_1} \otimes \hat{I}_{\mathcal{A}_2} \rangle, \quad \forall\, T \in \mathcal{A} \otimes_{min} \mathcal{B}.$$

If both ψ_1, ψ_2 are faithful traces of $\mathcal{A}_i$, then $\psi = \psi_1 \otimes_{min} \psi_2$ is also a faithful trace of $\mathcal{A} \otimes_{min} \mathcal{B}$ (for example see [**3**]). And,

$$(id_{\mathcal{A} \otimes_{min} \mathcal{B}}, \mathcal{H}_1 \otimes \mathcal{H}_2, \hat{I}_{\mathcal{A}_1} \otimes \hat{I}_{\mathcal{A}_2})$$

is a GNS representation of $(\mathcal{A} \otimes_{min} \mathcal{B}, \psi_1 \otimes_{min} \psi_2)$.

Using the discussion as above and following the same strategy as in Lemma 3.1.2, we can prove the following result, whose proof is skipped.

Lemma 5.2.1. *Suppose that $\mathcal{A}$ is a separable unital C^*-algebras with a faithful trace ψ. Let $\mathcal{H} = L^2(\mathcal{A}, \phi)$. Suppose that $\mathcal{B}$ is a finite dimensional C^*-algebras with a basis $1, b_1, \ldots, b_{d-1}$, where d is the complex dimension of $\mathcal{B}$. Suppose that $\{\tau, \tau_\gamma\}_{\gamma=1}^{\infty}$ is a family of faithful tracial states of $\mathcal{B}$ satisfying*

$$\lim_{\gamma\to\infty} \tau_\gamma(b) = \tau(b) \qquad \forall\ b \in \mathcal{B}.$$

Let $\mathbb{C}^d$ be a d-dimensional complex Hilbert space with an orthonormal basis $e_1, \ldots, e_d$. Then there is a sequence of faithful unital $*$-representations $\rho_\tau, \rho_{\tau_\gamma} : \mathcal{A} \otimes_{min} \mathcal{B} \to B(\mathcal{H}) \otimes_{min} \mathcal{M}_d(\mathbb{C})$ of $\mathcal{A} \otimes_{min} \mathcal{B}$ on $\mathcal{H} \otimes \mathbb{C}^d$ for $\gamma = 1, 2 \ldots$ such that

(i) *$(\rho_\tau, \mathcal{H} \otimes \mathbb{C}^d, \hat{I}_{\mathcal{A}} \otimes e_1)$ and $(\rho_{\tau_\gamma}, \mathcal{H} \otimes \mathbb{C}^d, \hat{I}_{\mathcal{A}} \otimes e_1)$ are GNS representations of $(\mathcal{A} \otimes_{min} \mathcal{B}, \psi \otimes_{min} \tau)$, and $(\mathcal{A} \otimes_{min} \mathcal{B}, \psi \otimes_{min} \tau_\gamma)$ respectively.*

(ii) *For each $1 \le i \le d-1$,*

$$\lim_{\gamma\to\infty} \|\rho_{\tau_\gamma}(a \otimes b_i) - \rho_\tau(a \otimes b_i)\| = 0, \qquad \forall\ a \in \mathcal{A}$$

The proof of the following result is similar to Lemma 3.1.3 and is skipped.

Lemma 5.2.2. *Suppose that $\mathcal{A}_i$, $i = 1, 2$, is a separable unital C^*-algebra with a faithful tracial state ψ_i. Suppose that $\mathcal{B}$ is a finite dimensional C^*-algebra with a family $\{\tau, \tau_\gamma\}_{\gamma=1}^{\infty}$ of faithful tracial states of $\mathcal{B}$ such that*

$$\lim_{\gamma\to\infty} \tau_\gamma(b) = \tau(b), \qquad \forall\ b \in \mathcal{B}.$$

Suppose that $x_1, \ldots, x_n$ is a family of elements in $(\mathcal{A}_1, \psi) *_{red} (\mathcal{B} \otimes_{min} \mathcal{A}_2, \tau \otimes_{min} \psi_2)$. Then, for any $\epsilon > 0$, there is a $\gamma_0 > 0$ such that

$$\{x_1, \ldots, x_n\} \subseteq_\epsilon (\mathcal{A}_1, \psi) *_{red} (\mathcal{B} \otimes_{min} \mathcal{A}_2, \tau_\gamma \otimes_{min} \psi_2), \qquad \forall\ \gamma > \gamma_0.$$

5.3. Some conclusions. Suppose that $(\mathcal{A}, \psi) \in \mathcal{S}$, where $\mathcal{S}$ is defined in Definition 5.1.1. Then $(\mathcal{A}, \psi) *_{red} (C_r^*(F_n), \tau_{F_n})$ is an MF algebra for all $n \ge 2$. Consider an action α of $\mathbb{Z}_n$ on $(\mathcal{A}, \psi) *_{red} (C_r^*(F_n), \tau_{F_n})$, induced by the following mapping: *if g is a natural generator of $\mathbb{Z}_n$ and $u_1, \ldots, u_n$ are the natural generators of $C_r^*(F_n)$, then*

$$\alpha(g)(x) = x, \qquad \forall\ x \in \mathcal{A};$$
$$\alpha(g)(u_i) = u_{i+1} \ for\ 1 \le i \le n-1;\ = u_1\ for\ i = n.$$

Using the same strategy as in the proof of Corollary 3.2.1, we have the following result.

Lemma 5.3.1. *Suppose that $(\mathcal{A}, \psi) \in \mathcal{S}$, where $\mathcal{S}$ is defined in Definition 5.1.1. Then for all $n \ge 2$,*

$$(\mathcal{A} \otimes_{min} C_r^*(\mathbb{Z}_n), \psi \otimes \tau_{\mathbb{Z}_n}) *_{red} (C_r^*(F_n), \tau_{F_n})$$

is an MF algebra.

Following the notation as above. Consider an action β of $\mathbb{Z}_n$ on

$$(\mathcal{A} \otimes_{min} C_r^*(\mathbb{Z}_n), \psi \otimes_{min} \tau_{\mathbb{Z}_n}) *_{red} (C_r^*(F_n), \tau_{F_n}),$$

induced by the following mapping: *if h is a natural generator of $\mathbb{Z}_n$, then*

$$\beta(h)(x) = x, \qquad \forall\, x \in \mathcal{A};$$
$$\beta(h)(v) = e^{2\pi i/n} v, \qquad \text{where } v \text{ is a natural generator of } C_r^*(\mathbb{Z}_n);$$
$$\beta(h)(u_j) = u_{j+1} \qquad \text{for } 1 \le j \le n-1;$$
$$\beta(h)(u_n) = u_1.$$

Modifying the proof of Lemma 3.2.3 slightly, we have the following result.

LEMMA 5.3.2. *Suppose that $(\mathcal{A}, \psi) \in \mathcal{S}$, where $\mathcal{S}$ is defined in Definition 5.1.1. Then for all $n \ge 2$,*

$$(\mathcal{A} \otimes_{min} \mathcal{M}_n(\mathbb{C}), \psi \otimes_{min} \tau_n) *_{red} (\mathcal{A} \otimes_{min} \mathcal{M}_n(\mathbb{C}), \psi \otimes_{min} \tau_n)$$

is a C^*-subalgebra of

$$\big((\mathcal{A} \otimes_{min} C_r^*(\mathbb{Z}_n), \psi \otimes \tau_{\mathbb{Z}_n}) *_{red} (C_r^*(F_n), \tau_{F_n})\big) \rtimes_{\beta,r} \mathbb{Z}_n;$$

and, therefore, is an MF algebra, where $\mathcal{M}_n(\mathbb{C})$ is $n \times n$ matrix algebra with a trace τ_n.

Combining Lemma 5.2.2, Lemma 5.3.2 and the strategy used in Theorem 3.3.1, we have the following result.

THEOREM 5.3.1. *Suppose that $(\mathcal{A}, \psi) \in \mathcal{S}$, where $\mathcal{S}$ is defined in Definition 5.1.1. Suppose that $\mathcal{B}_i$ is a unital AF algebra with a faithful trace ϕ_i for $i = 0, 1, 2, \ldots, n$. Then*

$$(\mathcal{A} \otimes_{min} \mathcal{B}_0, \psi \otimes_{min} \phi_0) *_{red} (\mathcal{A} \otimes_{min} \mathcal{B}_1, \psi \otimes_{min} \phi_1) *_{red} \cdots *_{red} (\mathcal{A} \otimes_{min} \mathcal{B}_n, \psi \otimes_{min} \phi_n)$$

is an MF algebra.

By Lemma 3.3.3, we have the following result.

THEOREM 5.3.2. *Suppose that $(\mathcal{A}, \psi) \in \mathcal{S}$, where $\mathcal{S}$ is defined in Definition 5.1.1. Suppose that $\mathcal{B}_i$ is a unital AH algebra with a faithful trace ϕ_i for $i = 0, 1, 2, \ldots, n$. Then*

$$(\mathcal{A} \otimes_{min} \mathcal{B}_0, \psi \otimes_{min} \phi_0) *_{red} (\mathcal{A} \otimes_{min} \mathcal{B}_1, \psi \otimes_{min} \phi_1) *_{red} \cdots *_{red} (\mathcal{A} \otimes_{min} \mathcal{B}_n, \psi \otimes_{min} \phi_n)$$

is an MF algebra. In particular, for every $n \ge 2$,

$$(\mathcal{A} \otimes_{min} \mathcal{B}_0, \psi \otimes_{min} \phi_0) *_{red} (C_r^*(F_n), \tau_{F_n})$$

is an MF algebra. I.e.

$$(\mathcal{A} \otimes_{min} \mathcal{B}_0, \psi \otimes_{min} \phi_0) \in \mathcal{S}.$$

COROLLARY 5.3.1. *For $i = 1, 2$, let $\mathcal{A}_1^{(i)}, \ldots, \mathcal{A}_n^{(i)}, \mathcal{B}^{(i)}$ be a family of unital AH algebras with faithful tracial states $\psi_1^{(i)}, \ldots, \psi_n^{(i)}, \phi^{(i)}$, respectively. Let*

$$(\mathcal{A}^{(i)}, \psi^{(i)}) = (\mathcal{A}_1^{(i)}, \psi_1^{(i)}) *_{red} \cdots *_{red} (\mathcal{A}_n^{(i)}, \psi_n^{(i)}), \qquad for\ i = 1, 2;$$

and $\psi^{(i)} \otimes_{min} \phi^{(i)}$ be a faithful trace on $\mathcal{A}^{(i)} \otimes_{min} \mathcal{B}^{(i)}$. Then

$$(\mathcal{A}^{(1)} \otimes_{min} \mathcal{B}^{(1)}, \psi^{(1)} \otimes_{min} \phi^{(1)}) *_{red} (\mathcal{A}^{(2)} \otimes_{min} \mathcal{B}^{(2)}, \psi^{(2)} \otimes \phi^{(2)})$$

is an MF algebra.

PROOF. Let

$$(\mathcal{A},\tau) = (\mathcal{A}^{(1)},\psi^{(1)}) *_{red} (\mathcal{A}^{(2)},\psi^{(2)}).$$

Let $\tau\otimes_{min}\phi^{(1)}\otimes_{min}\phi^{(2)}$ be a faithful tracial state on $\mathcal{A}\otimes_{min}\mathcal{B}_1\otimes_{min}\mathcal{B}_2$. By Theorem 3.3.3 and Theorem 5.3.2, we know that

$$(\mathcal{A},\tau)\in\mathcal{S};$$

and

$$\mathcal{D} = (\mathcal{A}\otimes_{min}\mathcal{B}_1\otimes_{min}\mathcal{B}_2, \tau\otimes_{min}\phi^{(1)}\otimes_{min}\phi^{(2)}) *_{red} (C_r^*(F_2),\tau_{F_2})$$

is an MF algebra. Therefore, embedded as a C*-subalgebra of $\mathcal{D}$,

$$(\mathcal{A}^{(1)}\otimes_{min}\mathcal{B}^{(1)},\psi^{(1)}\otimes_{min}\phi^{(1)}) *_{red} (\mathcal{A}^{(2)}\otimes_{min}\mathcal{B}^{(2)},\psi^{(2)}\otimes_{min}\phi^{(2)})$$

is an MF algebra. $\square$

EXAMPLE 5.3.1. *Suppose that $\mathcal{A}_i$, $i=1,2$, is an irrational C*-algebra, or a UHF algebra, with a faithful tracial state ψ_i. For all $m,n\geq 1$, let*

$$\mathcal{D} = (C_r^*(F_m)\otimes_{min}\mathcal{A}_1,\tau_{F_m}\otimes_{min}\psi_1) *_{red} (C_r^*(F_n)\otimes_{min}\mathcal{A}_2,\tau_{F_n}\otimes_{min}\psi_2).$$

Then $\mathcal{D}$ is an MF algebra and $Ext(\mathcal{D})$ is not a group.

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